Solutions to Problems

Getting Started

1 **(a)** This is shown in Figure SI.1.

	A	B	C	D
1	Economics Examination Marks			
2				
3	Candidate	Section A	Section B	
4	Fofaria	20	17	
5	Bull	38	12	
6	Eoin	34	38	
7	Arefin	40	52	
8	Cantor	29	34	
9	Devaux	30	49	
10				

Figure SI.1

(b) Type the heading Total Mark in cell D3. Type `=B4+C4` into cell D4. Click and drag down to D9.

(c) Type the heading Average: in cell C10. Type `=(SUM 4:D9))/6` in cell D10 and press Enter. [Note: Excel has lots of built-in functions for performing standard calculations such as this. To find the average you could just type `=AVERAGE(D4:D9)` in cell D10.]

(d) This is shown in Figure SI.2.

Economics Examination Marks

Candidate	Section A Mark	Section B Mark	Total Mark
Arefin	40	52	92
Bull	38	12	50
Cantor	29	34	63
Devaux	30	49	79
Eoin	34	38	72
Fofaria	20	17	37
		Average:	65.5

Figure SI.2

(e) Just put the cursor over cell C5 and type in the new mark of 42. Pressing the Enter key causes cells D5 and D10 to be automatically updated. The new spreadsheet is shown in Figure SI.3.

Economics Examination Marks

Candidate	Section A Mark	Section B Mark	Total Mark
Arefin	40	52	92
Bull	38	42	80
Cantor	29	34	63
Devaux	30	49	79
Eoin	34	38	72
Fofaria	20	17	37
		Average:	70.5

Figure SI.3

2 **(a)** 14

(b) 11

(c) 5

3 **(a)** 4; is the solution of the equation $2x - 8 = 0$.

(b) Figure SI.4 shows the graph of $2x - 8$ plotted between $x = 0$ and 10.

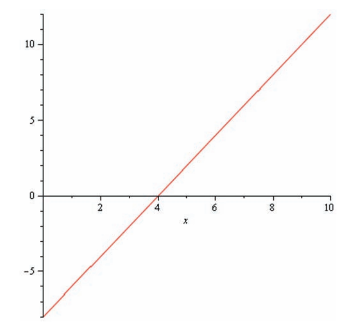


Figure SI.4

(c) $x^2 + 4x + 4$; the brackets have been 'multiplied out' in the expression $(x + 2)^2$.

(d) $7x + 4$; like terms in the expression $2x + 6 + 5x - 2$ have been collected together.

(e) Figure SI.5 shows the three-dimensional graph of the surface $x^3 - 3x + xy$ plotted between -2 and 2 in both the x and y directions.

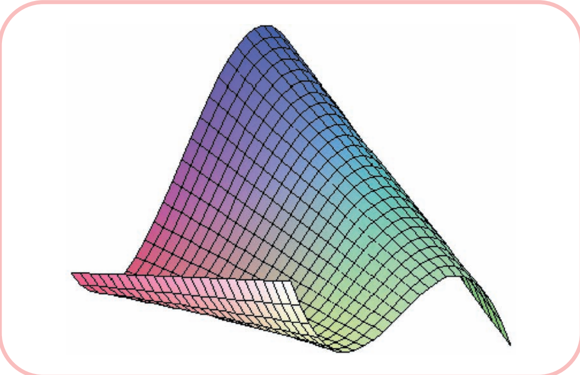


Figure SI.5

Chapter 1

Section 1.1

Practice Problems

1 (a) -30 (b) 2 (c) -5
(d) 5 (e) 36 (f) -1

2 (a) -1 (b) -7 (c) 5
(d) 0 (e) -91 (f) -5

3 (a) 19 (b) 1500 (c) 32 (d) 35

4 (a) $x + 9y$ (b) $2y + 4z$ (c) Not possible.
(d) $8r^2 + s + rs - 3s^2$ (e) $-4f$
(f) Not possible. (g) 0

5 (a) $5z - 2z^2$
(b) $6x - 6y + 3y - 6x = -3y$
(c) $x - y + z - x^2 - x + y = z - x^2$

6 (a) $7(d + 3)$ (b) $4(4w - 5q)$
(c) $3(3x - y + 3z)$ (d) $5Q(1 - 2Q)$

7 (a) $x^2 - 2x + 3x - 6 = x^2 + x - 6$
(b) $x^2 - xy + yx - y^2 = x^2 - y^2$
(c) $x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$
(d) $5x^2 - 5xy + 5x + 2yx - 2y^2 + 2y$
$= 5x^2 - 3xy + 5x - 2y^2 + 2y$

8 (a) $(x + 8)(x - 8)$
(b) $(2x + 9)(2x - 9)$

Exercise 1.1 (p. 30)

1 (a) -20 (b) 3 (c) -4 (d) 1
(e) -12 (f) 50 (g) -5 (h) 3
(i) 30 (j) 4

2 (a) -1 (b) -3 (c) -11 (d) 16
(e) -1 (f) -13 (g) 11 (h) 0
(i) -31 (j) -2

3 (a) -3 (b) 2 (c) 18 (d) -15
(e) -41 (f) -3 (g) 18 (h) -6
(i) -25 (j) -6

4 (a) $2PQ$ (b) $8I$ (c) $3xy$
(d) $4qwz$ (e) b^2 (f) $3k^2$

5 (a) $19w$ (b) $4x - 7y$ (c) $9a + 2b - 2c$
(d) $x^2 + 2x$ (e) $4c - 3cd$ (f) $2st + s^2 + t^2 + 9$

6 (a) 10 (b) 18 (c) 2000
(d) 96 (e) 70

7 (a) 1 (b) 5 (c) -6
(d) -6 (e) -30 (f) 44

8 (a) 16
(b) Presented with the calculation, -4^2, your calculator uses BIDMAS, so squares first to get 16 and then subtracts from zero to give a final answer, -16. To obtain the correct answer you need to use brackets:
[( ] [−] [4] [ )] [x^2] [=]

9 (a) 9 (b) 21; no

10 (a) 43.96 (b) 1.13 (c) 10.34 (d) 0.17
(e) 27.38 (f) 3.72 (g) 62.70 (h) 2.39

11 (a) $7x - 7y$ (b) $15x - 6y$ (c) $4x + 12$
(d) $21x - 7$ (e) $3x + 3y + 3z$ (f) $3x^2 - 4x$
(g) $-2x - 5y + 4z$

12 (a) $5(5c + 6)$ (b) $9(x - 2)$ (c) $x(x + 2)$
(d) $4(4x - 3y)$ (e) $2x(2x - 3y)$ (f) $5(2d - 3e + 10)$

13 (a) $x^2 + 7x + 10$ (b) $a^2 + 3a - 4$ (c) $d^2 - 5d - 24$
(d) $6s^2 + 23s + 21$ (e) $2y^2 + 5y + 3$ (f) $10t^2 - 11t - 14$
(g) $9n^2 - 4$ (h) $a^2 - 2ab + b^2$

14 (a) $6x + 2y$ (b) $11x^2 - 3x - 3$ (c) $14xy + 2x$;
(d) $6xyz + 2xy$ (e) $10a - 2b$ (f) $17x + 22y$
(g) $11 - 3p$ (h) $x^2 + 10x$

15 (a) $(x + 2)(x - 2)$ (b) $(Q + 7)(Q - 7)$
(c) $(x + y)(x - y)$ (d) $(3x + 10y)(3x - 10y)$

16 (a) $4x^2 + 8x - 2$ (b) $-13x$

Exercise 1.1* (p. 32)

1 (a) 3 (b) 5 (c) -7

2 (a) $2 - 7 - (9 + 3) = -17$ (b) $8 - (2 + 3) - 4 = -1$
(c) $7 - (2 - 6 + 10) = 1$

3 (a) -6 (b) 6 (c) -5 (d) -96
(e) -1 (f) 6 (g) $\frac{5}{4}$ (h) 63

4 (a) 6 (b) 2 (c) 5

5 $-y^2 + xy - 5x + 2y - 6$

6 (a) $2x - 2y$ (b) $2x$ (c) $-2x + 3y$

7 (a) $x^2 - 2x - 24$ (b) $6x^2 - 29x + 35$
(c) $6x^2 + 2xy - 4x$ (d) $12 - 2g + 3h - 2g^2 + gh$
(e) $2x - 2x^2 - 3xy + y - y^2$ (f) $a^2 - b^2 - c^2 - 2bc$

8 (a) $3(3x - 4y)$ (b) $x(x - 6)$
(c) $5x(2y + 3x)$ (d) $3xy(y - 2x + 4)$
(e) $x^2(x - 2)$ (f) $5xy^3(12x^3y^3 - 3xy + 4)$

9 (a) $(p + 5)(p - 5)$ (b) $(3c + 8)(3c - 8)$
(c) $2(4v + 5d)(4v - 5d)$ (d) $(4x^2 + y^2)(2x + y)(2x - y)$

10 (a) 112 600 000 (b) 1.7999
(c) 283 400 (d) 246 913 577

Section 1.2

Practice Problems

1 (a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{2y}$ (d) $\frac{1}{2+3x}$ (e) $\frac{1}{x-4}$

2 (a) $\frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8}$

(b) $\cancel{7} \times \frac{1}{\cancel{14}_2} = \frac{1}{2}$

(c) $\frac{2}{3} \div \frac{8}{9} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}^3}{\cancel{8}_4} = \frac{3}{4}$

(d) $\frac{8}{9} \div 16 = \frac{{}^1\cancel{8}}{9} \times \frac{1}{\cancel{16}_2} = \frac{1}{18}$

3 (a) $\frac{3}{7} - \frac{1}{7} = \frac{2}{7}$

(b) $\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$

(c) $\frac{7}{18} - \frac{1}{4} = \frac{14}{36} - \frac{9}{36} = \frac{5}{36}$

4 (a) $\frac{5}{\cancel{x-1}} \times \frac{\cancel{x-1}}{x+2} = \frac{5}{x+2}$

(b) $\frac{x^2}{x+10} \div \frac{x}{x+1} = \frac{x^{\cancel{2}}}{x+10} \times \frac{x+1}{\cancel{x}} = \frac{x(x+1)}{x+10}$

(c) $\frac{4}{x+1} + \frac{1}{x+1} = \frac{4+1}{x+1} = \frac{5}{x+1}$

(d) $\frac{2}{x+1} - \frac{1}{x+2}$

$= \frac{2(x+2)}{(x+1)(x+2)} - \frac{(1)(x+1)}{(x+1)(x+2)}$

$= \frac{(2x+4) - (x+1)}{(x+1)(x+2)} = \frac{(x+3)}{(x+1)(x+2)}$

5 (a) $4x + 5 = 5x - 7$
$5 = x - 7$ (subtract $4x$ from both sides)
$12 = x$ (add 7 to both sides)

(b) $3(3 - 2x) + 2(x - 1) = 10$
$9 - 6x + 2x - 2 = 10$ (multiply out brackets)
$7 - 4x = 10$ (collect like terms)
$-4x = 3$ (subtract 7 from both sides)
$x = -\frac{3}{4}$ (divide both sides by -4)

(c) $\frac{4}{x-1} = 5$
$4 = 5(x - 1)$ (multiply both sides by $x - 1$)
$4 = 5x - 5$ (multiply out brackets)
$9 = 5x$ (add 5 to both sides)
$\frac{9}{5} = x$ (divide both sides by 5)

(d) $\frac{3}{x} = \frac{5}{x-1}$
$3(x - 1) = 5x$ (cross-multiplication)
$3x - 3 = 5x$ (multiply out brackets)
$-3 = 2x$ (subtract $3x$ from both sides)
$-\frac{3}{2} = x$ (divide both sides by 2)

6 (a) $12 > 9$ (true) (b) $12 > 6$ (true)
(c) $3 > 0$ (true) (d) same as (c)
(e) $2 > 1$ (true) (f) $-24 > -12$ (false)
(g) $-6 > -3$ (false) (h) $2 > -1$ (false)
(i) $-4 > -7$ (true)

7 (a) $2x < 3x + 7$
$-x < 7$ (subtract $3x$ from both sides)
$x > -7$ (divide both sides by -1 changing sense because $-1 < 0$)

(b) $21x - 19 \geq 4x + 15$
$17x - 19 \geq 15$ (subtract $4x$ from both sides)
$17x \geq 34$ (add 19 to both sides)
$x \geq 2$ (divide both sides by 17, leaving inequality unchanged because $17 > 0$)

Exercise 1.2 (p. 47)

1 (a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{3}{5}$ (d) $\frac{1}{3}$ (e) $\frac{4}{3} = 1\frac{1}{3}$

2 (a) $\frac{2x}{3}$ (b) $\frac{1}{2x}$ (c) $\frac{1}{ac}$ (d) $\frac{2}{3xy}$ (e) $\frac{3a}{4b}$

3 (a) $\frac{p}{2q+3r}$ (b) $\frac{1}{x-4}$ (c) $\frac{b}{2a+1}$ (d) $\frac{2}{3-e}$

(e) $\frac{1}{x-2}$ (because $x^2 - 4 = (x + 2)(x - 2)$)

4 $\frac{x-1}{2x-2} = \frac{x-1}{2(x-1)} = \frac{1}{2}$; other two have no common factors on top and bottom.

5 (a) $\frac{3}{7}$ (b) $-\frac{1}{3}$ (c) $\frac{5}{6}$ (d) $\frac{7}{20}$

(e) $\frac{7}{18}$ (f) $\frac{5}{6}$ (g) $\frac{5}{8}$ (h) $\frac{2}{5}$

(i) $\frac{7}{12}$ (j) $\frac{1}{30}$ (k) $\frac{2}{27}$ (l) $\frac{21}{2} = 10\frac{1}{2}$

6 (a) $\frac{1}{x}$ (b) $\frac{2}{5}$ (c) $\frac{3x-2}{x^2}$ (d) $\frac{7y+2x}{xy}$

(e) 3 (f) $\frac{15c+10d}{36}$ (g) $\frac{x+2}{x+3}$ (h) $\frac{18h^2}{7}$

(i) $\frac{t}{20}$ (j) 1

7 (a) 5 (b) 6 (c) 18 (d) 2

(e) 10 (f) −1 (g) 60 (h) $1\frac{2}{3}$

(i) −5 (j) −3 (k) −2 (l) $-3\frac{2}{3}$

(m) $3\frac{1}{4}$ (n) 3 (o) $\frac{1}{4}$

8 (a), (d), (e), (f)

9 (a) $x > 1$ (b) $x \geq 3$ (c) $x \geq -3$ (d) $x > 2$

10 $\frac{2}{x^3}$

11 (a) $-\frac{7}{26}$ (b) $x \leq 10$

Exercise 1.2* (p. 48)

1 (a) $\frac{x-3}{2}$ (b) $\frac{3}{2x-1}$ (c) 4 (d) −1

(e) $\frac{1}{x-6}$ (f) $\frac{x+3}{x+4}$ (g) $\frac{1}{2x^2-5x+3}$

(h) $\frac{2x+5y}{3}$

2 (a) $\frac{5}{7}$ (b) $\frac{1}{10}$ (c) $\frac{3}{2}$ (d) $\frac{5}{48}$

(e) $\frac{8}{13}$ (f) $\frac{11}{9}$ (g) $\frac{141}{35}$ (h) $\frac{34}{5}$

(i) 6 (j) $\frac{7}{10}$ (k) $\frac{7}{9}$ (l) 4

3 (a) $x+6$ (b) $\frac{x+1}{x}$ or equivalently $1+\frac{1}{x}$

(c) $\frac{5}{xy}$ (d) $\frac{5x+2}{6}$ (e) $\frac{7x+3}{x(x+1)}$ (f) $\frac{3x+5}{x^2}$

(g) $\frac{x^2+x-2}{x+1}$ (h) $\frac{x+3}{x(x+1)}$

4 (a) $-\frac{11}{7}$ (b) 1 (c) $-\frac{35}{9}$ (d) 8

(e) $\frac{4}{5}$ (f) $\frac{1}{4}$ (g) $-\frac{11}{7}$ (h) 8

(i) 9 (j) $\frac{71}{21}$ (k) 7 (l) −9

(m) 1 (n) −5 (o) 3 (p) 5

5 $1.6 + \frac{5x}{7} = 6.75$; \$7.21

6 (a) \$3221.02 (b) \$60 000 (c) 10

7 (a) $x < -8\frac{3}{5}$ (b) $x > -\frac{12}{13}$ (c) $x \leq 13$

(d) $x > -3$ (e) $-1 < x \leq 3$

8 −3, −2, −1, 0.

9 (a) $\frac{3}{2x-1}$ (b) −1 (c) $x \geq \frac{11}{5}$

10 $\frac{x(x-1)}{2}$

Section 1.3

Practice Problems

1 From Figure S1.1 note that all five points lie on a straight line.

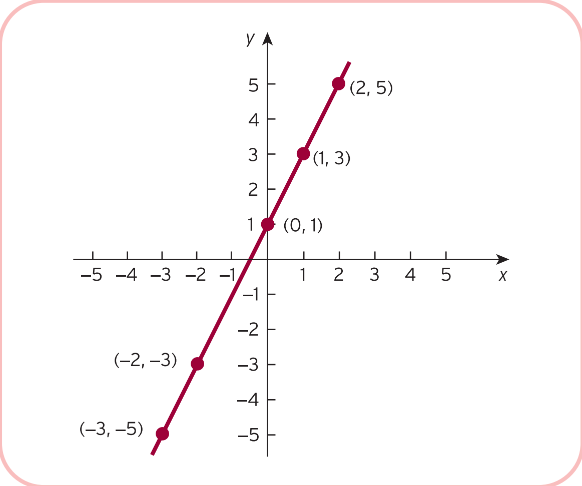


Figure S1.1

2

Point	Check	
(−1, 2)	2(−1) + 3(2) = −2 + 6 = 4	✓
(−4, 4)	2(−4) + 3(4) = −8 + 12 = 4	✓
(5, −2)	2(5) + 3(−2) = 10 − 6 = 4	✓
(2, 0)	2(2) + 3(0) = 4 + 0 = 4	✓

The graph is sketched in Figure S1.2.

The graph shows that (3, −1) does not lie on the line. This can be verified algebraically:

$$2(3) + 3(-1) = 6 - 3 = 3 \neq 4$$

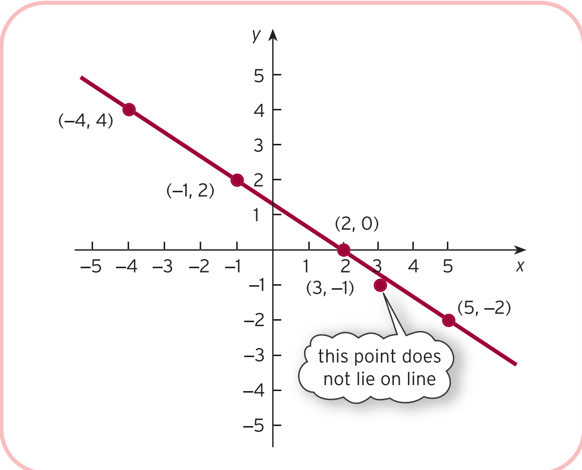


Figure S1.2

3 $3x - 2y = 4$

$3(2) - 2y = 4$ (substitute $x = -2$)

$6 - 2y = 4$

$-2y = -2$ (subtract 6 from both sides)

$y = 1$ (divide both sides by −2)

Hence (2, 1) lies on the line.

$3x - 2y = 4$

$3(-2) - 2y = 4$

$-6 - 2y = 4$ (substitute $x = 2$)

$-2y = 10$ (add 6 to both sides)

$y = -5$ (divide both sides by −2)

Hence (−2, −5) lies on the line.

The line is sketched in Figure S1.3.

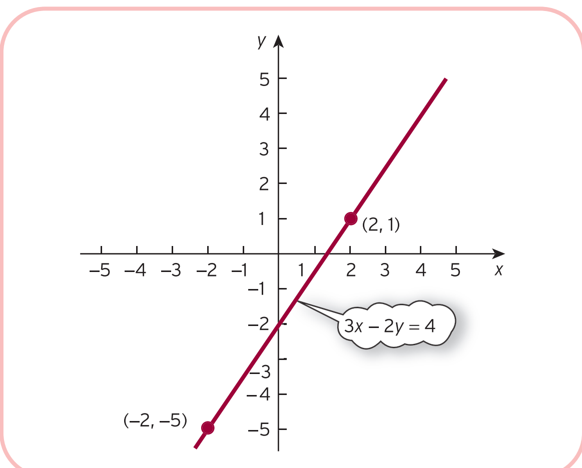


Figure S1.3

4 $x - 2y = 2$

$0 - 2y = 2$ (substitute $x = 0$)

$-2y = 2$

$y = -1$ (divide both sides by −2)

Hence (0, −1) lies on the line.

$x - 2y = 2$

$x - 2(0) = 2$ (substitute $y = 0$)

$x - 0 = 2$

$x = 2$

Hence (2, 0) lies on the line.

The graph is sketched in Figure S1.4.

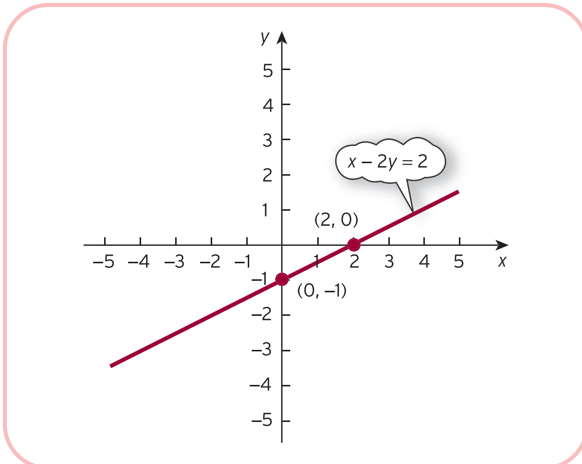


Figure S1.4

5 From Figure S1.5 the point of intersection is $(1, -\frac{1}{2})$.

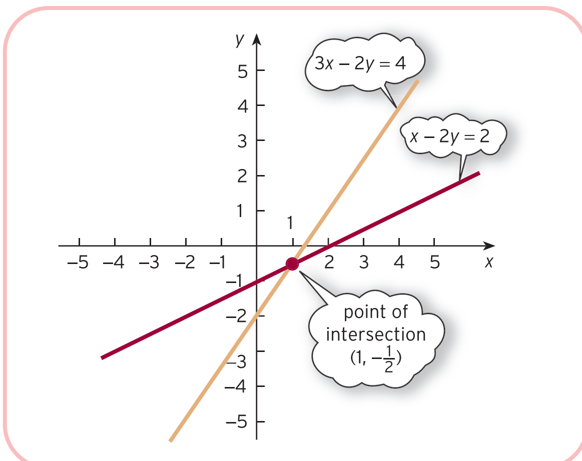


Figure S1.5

6 **(a)** $a = 1$, $b = 2$. The graph is sketched in Figure S1.6.

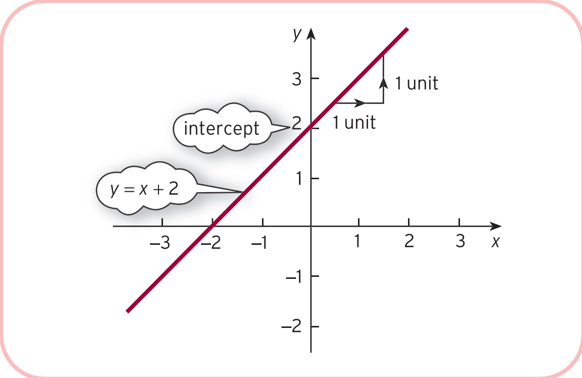


Figure S1.6

(b) $4x + 2y = 1$

$2y = 1 - 4x$ (subtract $4x$ from both sides)

$y = {}^1\!/_2 - 2x$ (divide both sides by 2)

so $a = -2$, $b = {}^1\!/_2$. The graph is sketched in Figure S1.7.

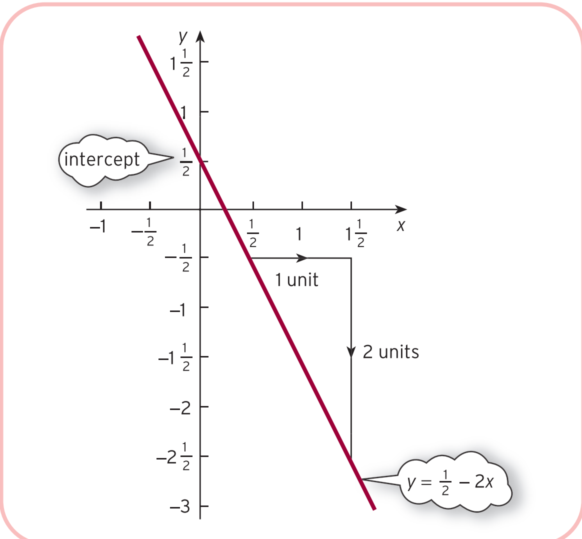


Figure S1.7

Exercise 1.3 (p. 64)

1 From Figure S1.8 the point of intersection is (2, 3).

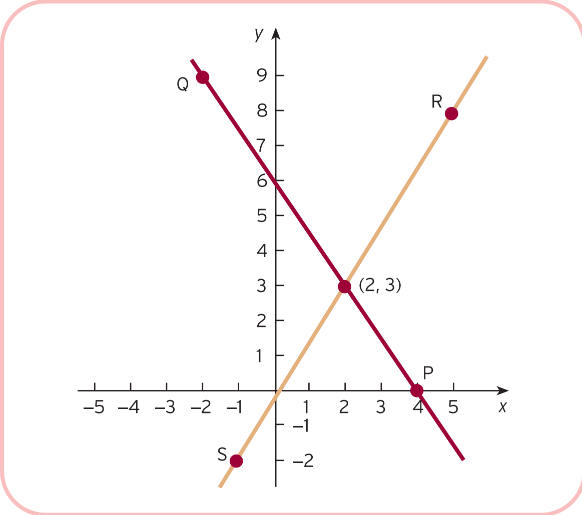


Figure S1.8

2 A, C, D, E

3 **(a)** 6 **(b)** -1; (6, 2), (1, -1)

4

x	y
0	8
6	0
3	4

The graph is sketched in Figure S1.9.

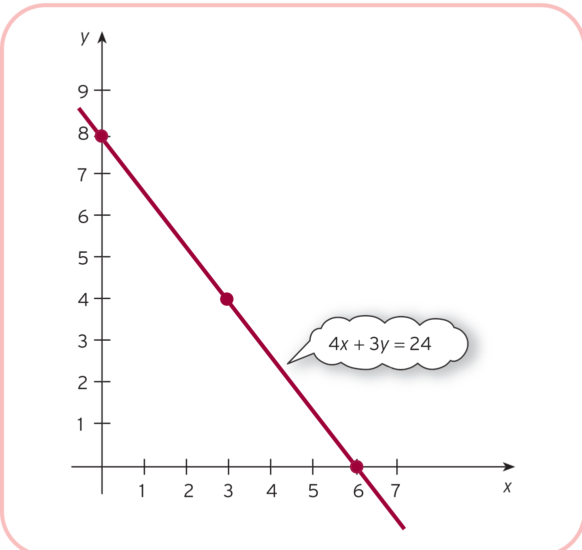


Figure S1.9

5 **(a)** $(-2, -2)$ **(b)** $(2, 1^1\!/_2)$ **(c)** $(1^1\!/_2, 1)$ **(d)** $(10, -9)$

6 **(a)** 5, 9 **(b)** 3, -1 **(c)** -1, 13 **(d)** 1, 4

(e) $-2, \dfrac{5}{2}$ **(f)** 5, -6

7 (a) The graph is sketched in Figure S1.10.

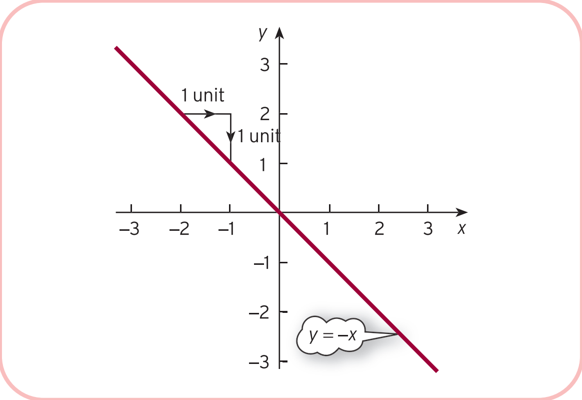


Figure S1.10

(b) The graph is sketched in Figure S1.11.

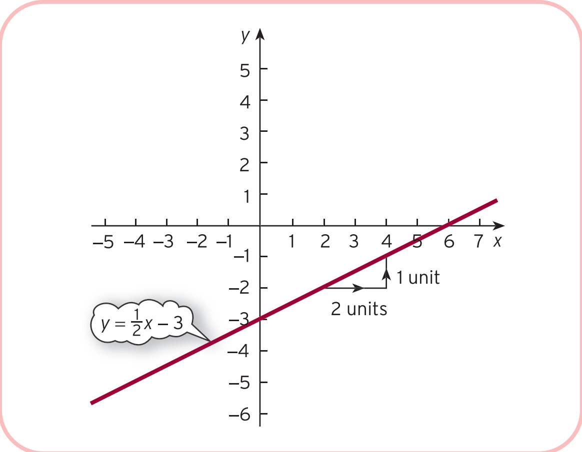


Figure S1.11

Exercise 1.3* (p. 65)

1 (5, –2), (10, 1), (0, –5)

2 **(a)** (2, 5) **(b)** (1, 4) **(c)** (–2, 3) **(d)** (–8, 3)

3 **(a)** 7, –34 **(b)** –1, 1 **(c)** $\frac{3}{2}$, –3

(d) 2, $\frac{5}{2}$ **(e)** $\frac{1}{5}$, 0 **(f)** 0, 2

(g) The vertical line, $x = 4$, has no gradient and does not intercept the y axis.

4 **(b)** and **(d)**

5 **(a)** $0.5x + 70$ **(b)** $x + 20$ **(c)** 100

6 **(1)** Gradients are $-\frac{a}{b}$ and $-\frac{d}{e}$ respectively so the lines are parallel when $\frac{a}{b} = \frac{d}{e}$ which gives $ae = bd$.

(2) Parallel lines so no solution.

7 $\left(0, \frac{c}{b}\right), \left(\frac{c}{a}, 0\right)$

Section 1.4

Practice Problems

1 **(a)** Step 1

It is probably easiest to eliminate y. This can be done by subtracting the second equation from the first:

$$\begin{aligned} 3x - 2y &= 4 \\ x - 2y &= 2\ - \\ \hline 2x \quad &= 2 \end{aligned}$$

Step 2

The equation $2x = 2$ has solution $x = 2/2 = 1$.

Step 3

If this is substituted into the first equation then

$$\begin{aligned} 3(1) - 2y &= 4 \\ 3 - 2y &= 4 \\ -2y &= 1 \quad \text{(subtract 3 from both sides)} \\ y &= -{}^1\!/_2 \quad \text{(divide both sides by } -2) \end{aligned}$$

Step 4

As a check the second equation gives

$$\begin{aligned} x - 2y &= 1 - 2(-1/2) \\ &= 1 - (-1) = 2 \quad ✓ \end{aligned}$$

Hence the solution is $x = 1$, $y = -{}^1\!/_2$.

If you decide to eliminate x then the corresponding steps are as follows:

Step 1

Triple the second equation and subtract from the first:

$$\begin{aligned} 3x - 2y &= 4 \\ 3x - 6y &= 6\ - \\ \hline 4y &= -2 \end{aligned}$$

Step 2

The equation $4y = -2$ has solution $y = -2/4 = -{}^1\!/_2$.

Step 3

If this is substituted into the first equation then

$$\begin{aligned} 3x - 2(-{}^1\!/_2) &= 4 \\ 3x + 1 &= 4 \\ 3x &= 3 \end{aligned}$$

(subtract 1 from both sides)

$$x = 1$$

(divide both sides by 3)

(b) Step 1

It is immaterial which variable is eliminated. To eliminate x multiply the first equation by 5, multiply the second by 3 and add:

$$\begin{aligned} 15x + 25y &= 95 \\ \underline{-15x + 6y} &\underline{= -33} + \\ 31y &= 62 \end{aligned}$$

Step 2

The equation $31y = 62$ has solution $y = 62/31 = 2$.

Step 3

If this is substituted into the first equation then

$$3x + 5(2) = 19$$

$$3x + 10 = 19$$

$$3x = 9$$

(subtract 10 from both sides)

$$x = 3$$

(divide both sides by 3)

Step 4

As a check the second equation gives

$$-5x + 2y = -5(3) + 2(2)$$

$$= -15 + 4 = -11 \quad ✓$$

Hence the solution is $x = 3$, $y = 2$.

2 (a) Step 1

To eliminate x multiply the first equation by 4, multiply the second equation by 3 and add:

$$\begin{aligned} 12x + 24y &= -8 \\ \underline{-12x + 24y} &\underline{= -3} + \\ 0y &= -11 \end{aligned}$$

Step 2

This is impossible, so there are no solutions.

(b) Step 1

To eliminate x multiply the first equation by 2 and add to the second:

$$\begin{aligned} -10x + 2y &= 8 \\ \underline{10x - 2y} &\underline{= -8} + \\ 0y &= 0 \end{aligned}$$

Step 2

This is true for any value of y, so there are infinitely many solutions.

3 Step 1

To eliminate x from the second equation multiply equation (2) by 2 and subtract from equation (1):

$$\begin{aligned} 2x + 2y - 5z &= -5 \\ \underline{2x - 2y + 2z} &\underline{= 6} - \\ 4y - 7z &= -11 \qquad (4) \end{aligned}$$

To eliminate x from the third equation multiply equation (1) by 3, multiply equation (3) by 2 and add:

$$\begin{aligned} 6x + 6y - 15z &= -15 \\ \underline{-6x + 2y + 4z} &\underline{= -4} + \\ 8y - 11z &= -19 \qquad (5) \end{aligned}$$

The new system is

$$\begin{aligned} 2x + 2y - 5z &= -5 \qquad (1) \\ 4y - 7z &= -11 \qquad (4) \\ 8y - 11z &= -19 \qquad (5) \end{aligned}$$

Step 2

To eliminate y from the third equation multiply equation (4) by 2 and subtract equation (5):

$$\begin{aligned} 8y - 14z &= -22 \\ \underline{8y - 11z} &\underline{= -19} \\ -3z &= -3 \qquad (6) \end{aligned}$$

The new system is

$$\begin{aligned} 2x + 2y - 5z &= -5 \qquad (1) \\ 4y - 7z &= -11 \qquad (4) \\ -3z &= -3 \qquad (6) \end{aligned}$$

Step 3

Equation (6) gives $z = -3/-3 = 1$. If this is substituted into equation (4) then

$$4y - 7(1) = -11$$

$$4y - 7 = -11$$

$$4y = -4 \quad \text{(add 7 to both sides)}$$

$$y = -1 \quad \text{(divide both sides by 4)}$$

Finally, substituting $y = -1$ and $z = 1$ into equation (1) produces

$$2x + 2(-1) - 5(1) = -5$$

$$2x - 7 = -5$$

$$2x = 2$$

(add 7 to both sides)

$$x = 1$$

(divide both sides by 2)

Step 4

As a check the original equations (1), (2) and (3) give

$$2(1) + 2(-1) - 5(1) = -5 \quad ✓$$

$$1 - (-1) + 1 = 3 \quad ✓$$

$$-3(1) + (-1) + 2(1) = -2 \quad ✓$$

Hence the solution is $x = 1$, $y = -1$, $z = 1$.

Exercise 1.4 (p. 77)

1 **(a)** $x = -2, y = -2$

(b) $x = 2, y = 3/2$

(c) $x = 3/2, y = 1$

(d) $x = 10, y = -9$

2 The lines are sketched in Figure S1.12.

(a) Infinitely many. **(b)** No solution.

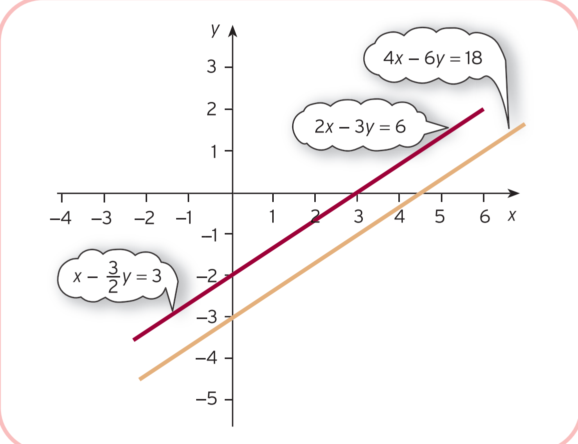


Figure S1.12

3 **(a)** Infinitely many. **(b)** No solution.

4 $k = -1$

Exercise 1.4* (p. 78)

1 **(a)** $x = 2, y = 5$ **(b)** $x = 1, y = 4$
(c) $x = -2, y = 3$ **(d)** $x = -8, y = 3$

2 **(a)** $a = 4, b = 8$ **(b)** $a = -3, b \neq \frac{1}{2}$

3 **(a)** $x = 3, y = -2, z = -1$

(b) $x = -1, y = 3, z = 4$

4 **(a)** No solution.

(b) Infinitely many solutions.

5 $k = 6$; no solutions otherwise.

Section 1.5

Practice Problems

1 **(a)** 0 **(b)** 48 **(c)** 16
(d) 25 **(e)** 1 **(f)** 17

The function g reverses the effect of f and takes you back to where you started. For example, if 25 is put into the function f, the outgoing number is 0; and when 0 is put into g, the original number, 25, is produced. We describe this by saying that g is the inverse of f (and vice versa).

2 The demand curve that passes through (0, 75) and (25, 0) is sketched in Figure S1.13. From this diagram we see that

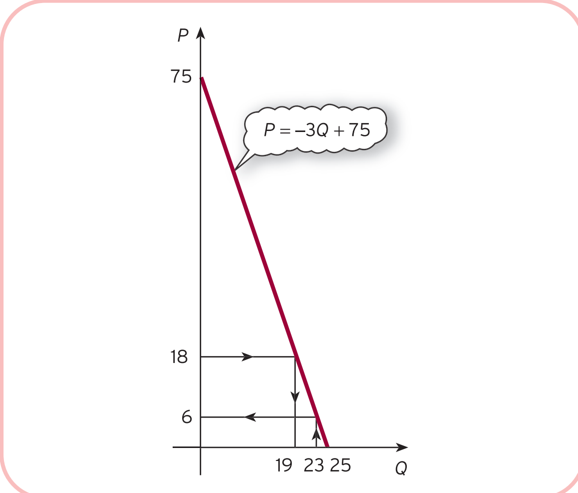


Figure S1.13

(a) $P = 6$ when $Q = 23$

(b) $Q = 19$ when $P = 18$

Alternatively, using algebra:

(a) Substituting $Q = 23$ gives

$$P = -3(23) + 75 = 6$$

(b) Substituting $P = 18$ gives $18 = -3Q + 75$ with solution $Q = 19$

3 **(a)** In equilibrium, $Q_S = Q_D = Q$, so

$$P = -4Q + 120$$

$$P = {}^{1}\!/_{3}Q + 29$$

Hence

$$-4Q + 120 = {}^{1}\!/_{3}Q + 29$$

(since both sides equal P)

$$-4{}^{1}\!/_{3}Q + 120 = 29$$

(subtract ${}^{1}\!/_{3}Q$ from both sides)

$$-4{}^{1}\!/_{3}Q = -91$$

(subtract 120 from both sides)

$$Q = 21$$

(divide both sides by $-4{}^{1}\!/_{3}$)

Substituting this value into either the demand or supply equations gives $P = 36$.

(b) After the imposition of a \$13 tax the supply equation becomes

$$P - 13 = {}^{1}\!/_{3}Q_S + 29$$

$$P = {}^{1}\!/_{3}Q_S + 42$$

(add 13 to both sides)

The demand equation remains unchanged, so, in equilibrium,

$P = -4Q + 120$

$P = \frac{1}{3}Q + 42$

Hence

$-4Q + 120 = \frac{1}{3}Q + 42$

This equation can now be solved as before to get $Q = 18$ and the corresponding price is $P = 48$. The equilibrium price rises from \$36 to \$48, so the consumer pays an additional \$12. The remaining \$1 of the tax is paid by the firm.

4 For good 1, $Q_{D_1} = Q_{S_1} = Q_1$ in equilibrium, so the demand and supply equations become

$Q_1 = 40 - 5P_1 + P_2$

$Q_1 = -3 + 4P_1$

Hence

$40 - 5P_1 - P_2 = -3 + 4P_1$

(since both sides equal Q_1)

$40 - 9P_1 - P_2 = -3$

(subtract $4P_1$ from both sides)

$-9P_1 - P_2 = -43$

(subtract 40 from both sides)

For good 2, $Q_{D_2} = Q_{S_2} = Q_2$ in equilibrium, so the demand and supply equations become

$Q_2 = 50 - 2P_1 + 4P_2$

$Q_2 = -7 + 3P_2$

Hence

$50 - 2P_1 - 4P_2 = -7 + 3P_2$

(since both sides equal Q_2)

$50 - 2P_1 - 7P_2 = -7$

(subtract $3P_2$ from both sides)

$-2P_1 - 7P_2 = -57$

(subtract 50 from both sides)

The equilibrium prices therefore satisfy the simultaneous equations

$-9P_1 - P_2 = -43$ (1)

$-2P_1 - 7P_2 = -57$ (2)

Step 1

Multiply equation (1) by 2 and (2) by 9 and subtract to get

$61P_2 = 427$ (3)

Step 2

Divide both sides of equation (3) by 61 to get $P_2 = 7$.

Step 3

Substitute P_2 into equation (1) to get $P_1 = 4$.

If these equilibrium prices are substituted into either the demand or the supply equations then $Q_1 = 13$ and $Q_2 = 14$.

The goods are complementary because the coefficient of P_2 in the demand equation for good 1 is negative, and likewise for the coefficient of P_1 in the demand equation for good 2.

Exercise 1.5 (p. 94)

1 **(a)** 21 **(b)** 45 **(c)** 15 **(d)** 2
(e) 10 **(f)** 0; inverse

2 The supply curve is sketched in Figure S1.14.

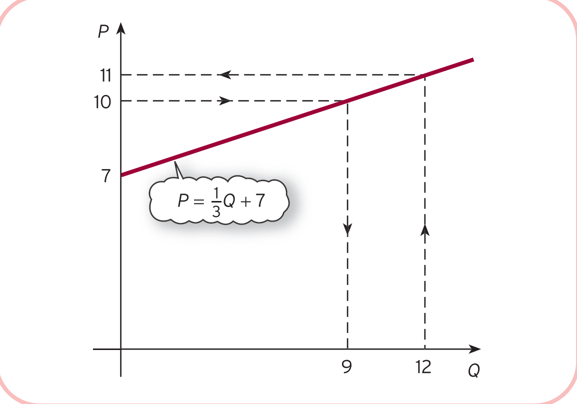


Figure S1.14

(a) 11 **(b)** 9

(c) 0; once the price falls below 7 the firm does not plan to produce any goods.

3 **(a)** Demand is 173. Additional advertising expenditure is 12.

(b) Superior.

4 **(a)** 53

(b) Substitutable; a rise in P_A leads to an increase in Q.

(c) 6

5 **(a)** 20, 10, 45; line passes through these three points.

(b) Line passing through (50, 0) and (0, 50)

$Q = 20, P = 30$

(c) Price increases; quantity increases.

6 6

7 $P_1 = 40, P_2 = 10; Q_1 = 30, Q_2 = 55$

8 **(a)** $Q = 30$

(b) Substitutable; e.g. since coefficient of P_r is positive.

(c) $P = 14$

(d) **(i)** slope = –20, intercept = 135

(ii) slope = $-\frac{1}{20}$, intercept = 6.75

9 Superior; graph of **(b)** lies above that of **(a)**.

Substitutable; graph of **(c)** lies below that of **(a)**.

Exercise 1.5* (p. 96)

1 **(a)** As P_S rises, consumers are likely to switch to the good under consideration, so demand for this good also rises: that is, the graph shifts to the right.

(b) As P_C rises, demand for the bundle of goods as a whole is likely to fall, so the graph shifts to the left.

(c) Assuming that advertising promotes the good and is successful, demand rises and the graph shifts to the right. For some goods, such as drugs, advertising campaigns are intended to discourage consumption, so the graph shifts to the left.

2 $m = -\frac{3}{2}$, $c = 9$

3 0 and 30

4 **(1)** $P = 30$, $Q = 10$

(2) New supply equation is $0.85P = 2Q_S + 10$; $P = 33.6$, $Q = 9.28$.

5 **(a)** 17, 9 **(b)** \$324

6 $P_1 = 20$, $P_2 = 5$, $P_3 = 8$; $Q_1 = 13$, $Q_2 = 16$, $Q_3 = 11$

7 Change supply equation to $P = 2Q_S + 40 + t$.

In equilibrium

$$-3Q + 60 = 2Q + 40 + t$$
$$-5Q = -20 + t$$
$$Q = 4 - \frac{t}{5}$$

Substitute to get $P = 48 + \frac{3}{5}t$.

(a) $t = 5$ firm pays \$2

(b) $P = 45$, $Q = 5$

8 \$180, \$200, $\frac{1}{3}$.

(a) $\frac{1}{2}$ **(b)** $\frac{3}{5}$ **(c)** $\frac{2}{3}$; fraction is $\frac{k}{k+2}$

When $k = 6$, the fraction is $^3/_4$ so the consumer pays \$45.

[In general, if the supply and demand equations are

$$P = -aQ_D + b$$
$$P = cQ_S + d$$

the fraction of tax paid by the consumer is

$\frac{a}{a+d}$.]

Section 1.6

Practice Problems

1 **(a)** $\frac{1}{2}Q = 4$ (subtract 13 from both sides)

$Q = 8$ (multiply both sides by 2)

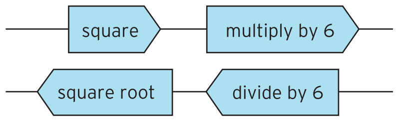
(b) $\frac{1}{2}Q = P - 13$ (subtract 13 from both sides)

$Q = 2(P - 13)$ (multiply both sides by 2)

$Q = 2P - 26$ (multiply out brackets)

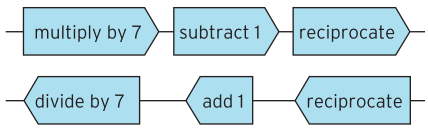
(c) $Q = 2 \times 17 - 26 = 8$

2 **(a)**



$$6x^2 = y$$
$$x^2 = \frac{y}{6} \quad \text{(divide both sides by 6)}$$
$$x = \sqrt{\frac{y}{6}} \quad \text{(square root both sides)}$$

(b)



$$\frac{1}{7x - 1} = y$$
$$7x - 1 = \frac{1}{y} \quad \text{(reciprocate both sides)}$$
$$7x = \frac{1}{y} + 1 \quad \text{(add 1 to both sides)}$$
$$x = \frac{1}{7}\left(\frac{1}{y} + 1\right) \quad \text{(divide both sides by 7)}$$

3 **(a)** $x - ay = cx + y$

$x = cx + y + ay$

(add ay to both sides)

$x - cx = y + ay$

(subtract cx from both sides)

$(1 - c)x = (1 + a)y$

(factorize both sides)

$$x = \left(\frac{1 + a}{1 - c}\right)y$$

(divide both sides by $1 - c$)

(b) $$y = \frac{x-2}{x+4}$$

$(x+4)y = x - 2$

(multiply both sides by $x + 4$)

$xy + 4y = x - 2$

(multiply out the brackets)

$xy = x - 2 - 4y$

(subtract $4y$ from both sides)

$xy - x = -2 - 4y$

(subtract x from both sides)

$(y-1)x = -2 - 4y$

(factorize left-hand side)

$$x = \frac{-2-4y}{y-1}$$

(divide both sides by $y - 1$)

Exercise 1.6 (p. 106)

1 $Q = \frac{1}{2}P - 4$; 22

2 (a) $y = 2x + 5$ (b) $y = 2(x+5)$

(c) $y = \frac{5}{x^2}$ (d) $y = 2(x+4)^2 - 3$

3 (a) multiply by 5 → add 3

(b) add 3 → multiply by 5

(c) multiply by 6 → subtract 9

(d) square → multiply by 4 → subtract 6

(e) divide by 2 → add 7

(f) reciprocate → multiply by 2

(g) add 3 → reciprocate

4 (a) $x = \frac{1}{9}(y+6)$ (b) $x = 3y - 4$ (c) $x = 2y$

(d) $x = 5(y-8)$ (e) $x = \frac{1}{y} - 2$ (f) $x = \frac{1}{3}\left(\frac{4}{y} + 7\right)$

5 (a) $P = \frac{Q}{a} - \frac{b}{a}$ (b) $Y = \frac{b+1}{1-a}$ (c) $P = \frac{1}{aQ} - \frac{b}{a}$

6 $x = \frac{3}{y+2}$

Exercise 1.6* (p. 107)

1 (1)

(a) multiply by 9 → add 1

(b) multiply by -1 → add 3

(c) square → multiply by 5 → subtract 8

(d) multiply by 3 → add 5 → square root

(e) square → add 8 → reciprocate → multiply by 4

(2) (a) $x = \frac{y-1}{9}$ (b) $x = 3 - y$ (c) $x = \pm\sqrt{\frac{y+8}{5}}$

(d) $x = \frac{y^2-5}{3}$ (e) $x = \pm\sqrt{\frac{4}{y} - 8}$

2 (a) $x = \frac{c-a}{b}$ (b) $x = \frac{a^2-b}{a+1}$ (c) $x = (g-e)^2 - f$

(d) $x = \frac{ma^2}{b^2} + n$ (e) $x = \frac{n^2}{m^2} + m$ (f) $x = \left(\frac{a^2+b^2}{b-a}\right)^2$

3 $t = \frac{V+1}{V-5}$; 11

4 $r = 100\left(\sqrt[n]{\frac{S}{P}} - 1\right)$

5 (a) $G = Y(1 - a + at) + aT - b - I$

(b) $$T = \frac{G + b + I - Y(1 - a + at)}{a}$$

(c) $$t = \frac{G + b + I - Y + aY - aT}{aY}$$

(d) $$a = \frac{G + b + I - Y}{T - Y + tY}$$

Section 1.7

Practice Problems

1 $S = Y - C$

$= Y - (0.8Y + 25)$ (substitute expression for C)

$= Y - 0.8Y - 25$ (multiply out brackets)

$= 0.2Y - 25$ (collect terms)

2 $Y = C + I$ (from theory)

$C = 0.8Y + 25$ (given in question)

$I = 17$ (given in question)

Substituting the given value of I into the first equation gives

$Y = C + 17$

and if the expression for C is substituted into this then

$Y = 0.8Y + 42$

$0.2Y = 42$ (subtract $0.8Y$ from both sides)

$Y = 210$ (divide both sides by 0.2)

Repeating the calculations with $I = 18$ gives $Y = 215$, so a 1 unit increase in investment leads to a 5 unit increase in income. The scale factor, 5, is called the investment multiplier. In general, the investment multiplier is given by $1/(1 - a)$, where a is the marginal propensity to consume. The foregoing is a special case of this with $a = 0.8$.

3 $Y = C + I + G$ (1)

$G = 40$ (2)

$I = 55$ (3)

$C = 0.8Y_d + 25$ (4)

$T = 0.1Y + 10$ (5)

$Y_d = Y - T$ (6)

Substituting equations (2) and (3) into equation (1) gives

$Y = C + 95$ (7)

Substituting equation (5) into (6) gives

$Y_d = Y - (0.1Y + 10)$

$= 0.9Y - 10$

so, from equation (4),

$C = 0.8(0.9Y - 10) + 25$

$= 0.72Y + 17$ (8)

Finally, substituting equation (8) into (7) gives

$Y = 0.72Y + 112$

which has solution $Y = 400$.

4 The commodity market is in equilibrium when

$Y = C + I$

so we can substitute the given expressions for consumption ($C = 0.7Y + 85$) and investment ($I = 50r + 1200$) to deduce that

$Y = 0.7Y - 50r + 1285$

which rearranges to give the IS schedule,

$0.3Y + 50r = 1285$ (1)

The money market is in equilibrium when

$M_S = M_D$

Now we are given that $M_S = 500$ and that total demand,

$M_D = L_1 + L_2 = 0.2Y - 40r + 230$

so that

$500 = 0.2Y - 40r + 230$

which rearranges to give the LM schedule,

$0.2Y - 40r = 270$ (2)

We now solve equations (1) and (2) as a pair of simultaneous equations.

Step 1

Multiply equation (1) by 0.2 and (2) by 0.3 and subtract to get

$22r = 176$

Step 2

Divide through by 22 to get $r = 8$.

Step 3

Substitute $r = 8$ into equation (1) to give $Y = 2950$.

The IS and LM curves shown in Figure S1.15 confirm this, since the point of intersection has coordinates (8, 2950). A change in I does not affect the LM schedule. However, if the autonomous level of investment increases from its current level of 1200 then the right-hand side of the IS schedule (1) will rise. The IS curve moves upwards, causing both r and Y to increase.

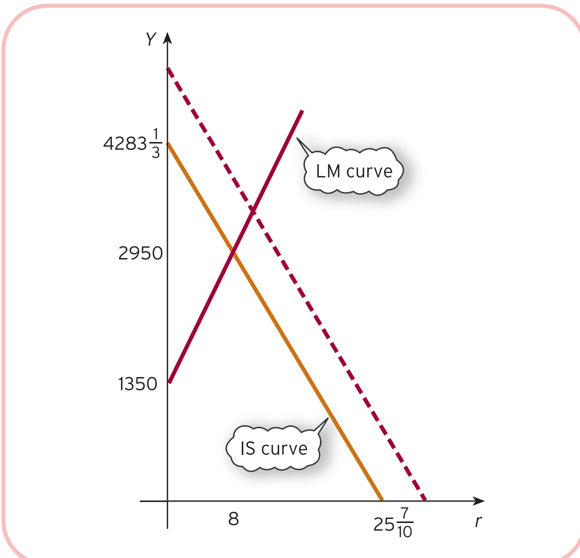


Figure S1.15

Exercise 1.7 (p. 123)

1 (a) 40 (b) 0.7; $Y = \frac{10}{7}(C - 40)$; 100

2 (a) $S = 0.1Y - 72$ (b) $S = 0.2Y - 100$

3 (a) 325 (b) 225 (c) 100

4 $10a + b = 28$

$30a + b = 44$

$a = 0.8, b = 20; \quad Y = 165$

5 187.5

Exercise 1.7* (p. 124)

1 **(a)** $S = 0.3Y - 30$

(b) $S = \frac{10Y - 500}{Y + 10}$

2 $C = \frac{aI^* + b}{1 - a}$

3 $a = \frac{Y - b - I^*}{Y}$

4 825

5 $Y = 2500, r = 10$

6 $C = a(Y - 20) + 50 = aY - 20a + 50$

$Y = aY - 20a + 74$

$(1 - a)Y = 74 - 20a \quad \Rightarrow \quad Y = \frac{74 - 20a}{1 - a}$

$a = \frac{Y - 74}{Y - 20}; \quad a = 0.6, C = 131$

7 **(a)** $C = 120 + 0.8Y$

(b) $C = 40 + 0.8Y$

(c) $C = 120 + 0.6Y$

With a lump sum tax, the graph has the same slope but has been shifted downwards.

With a proportional tax, the graph has the same intercept but is less steep.

(a) 600 **(b)** 200 **(c)** 300

8 $0.9Y + 30$; 300

(a) Slope decreases; 150.

(b) Shifts up 5 units; 350.

Chapter 2

Section 2.1

Practice Problems

1 **(a)**

$$x^2 - 100 = 0$$
$$x^2 = 100$$
$$x = \pm\sqrt{100}$$
$$x = \pm 10$$

(b)

$$2x^2 - 8 = 0$$
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x = \pm 2$$

(c)

$$x^2 - 3 = 0$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$
$$x = \pm 1.73 \quad \text{(to 2 decimal places)}$$

(d)

$$x^2 - 5.72 = 0$$
$$x^2 = 5.72$$
$$x = \pm\sqrt{5.72}$$
$$x = \pm 2.39 \quad \text{(to 2 decimal places)}$$

(e)

$$x^2 + 1 = 0$$
$$x^2 = -1$$

This equation does not have a solution, because the square of a number is always positive. Try using your calculator to find $\sqrt{(-1)}$. An error message should be displayed.

(f)

$$3x^2 + 6.21 = 0$$
$$3x^2 = -6.21$$
$$x^2 = -2.07$$

This equation does not have a solution, because it is impossible to find the square root of a negative number.

(g) $x^2 = 0$

This equation has exactly one solution, $x = 0$.

2 **(a)** $a = 2, b = -19, c = -10$

$$x = \frac{-(-19) \pm \sqrt{((-19)^2 - 4(2)(-10))}}{2(2)}$$

$$= \frac{19 \pm \sqrt{(361 + 80)}}{4}$$

$$= \frac{19 \pm \sqrt{441}}{4} = \frac{19 \pm 21}{4}$$

This equation has two solutions:

$$x = \frac{19 + 21}{4} = 10$$

$$x = \frac{19 - 21}{4} = -\frac{1}{2}$$

(b) $a = 4, b = 12, c = 9$

$$x = \frac{-12 \pm \sqrt{((12)^2 - 4(4)(9))}}{2(4)}$$

$$= \frac{-12 \pm \sqrt{(144 - 144)}}{8}$$

$$= \frac{-12 \pm 0}{8}$$

This equation has one solution, $x = -^3/_2$.

(c) $a = 1, b = 1, c = 1$

$$x = \frac{-1 \pm \sqrt{((1)^2 - 4(1)(1))}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{(1-4)}}{2}$$

$$= \frac{-1 \pm \sqrt{(-3)}}{2}$$

This equation has no solutions, because $\sqrt{(-3)}$ does not exist.

(d) We first need to collect like terms to convert

$$x^2 - 3x + 10 = 2x + 4$$

into the standard form

$$ax^2 + bx + c = 0$$

Subtracting $2x + 4$ from both sides gives

$$x^2 - 5x + 6 = 0$$

$$a = 1, b = -5, c = 6$$

$$x = \frac{-(-5) \pm \sqrt{((5)^2 - 4(1)(6))}}{2(1)}$$

$$= \frac{5 \pm \sqrt{(25-24)}}{2}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5 \pm 1}{2}$$

This equation has two solutions:

$$x = \frac{5+1}{2} = 3$$

$$x = \frac{5-1}{2} = 2$$

3 (a) If $(x - 4)(x + 3) = 0$ then either

$x - 4 = 0$ with solution $x = 4$

or

$x + 3 = 0$ with solution $x = -3$

This equation has two solutions, $x = 4$ and $x = 3$.

(b) If $x(10 - 2x) = 0$ then either

$x = 0$

or

$10 - 2x = 0$ with solution $x = 5$

This equation has two solutions, $x = 0$ and $x = 5$.

(c) If $(2x - 6)(2x - 6) = 0$ then

$2x - 6 = 0$ with solution $x = 3$

This equation has one solution, $x = 3$.

4 (a)

x	−1	0	1	2	3	4
$f(x)$	21	5	−3	−3	5	21

The graph is sketched in Figure S2.1.

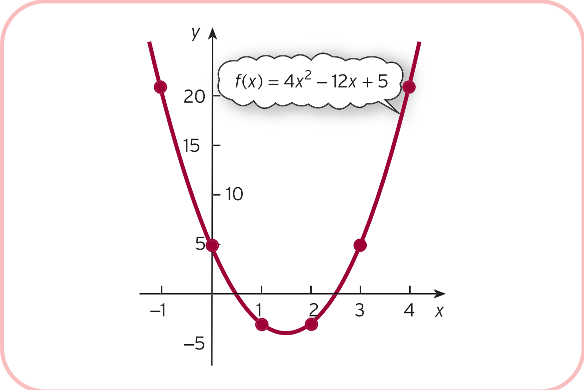


Figure S2.1

(b)

x	0	1	2	3	4	5	6
$f(x)$	−9	−4	−1	0	−1	−4	−9

The graph is sketched in Figure S2.2.

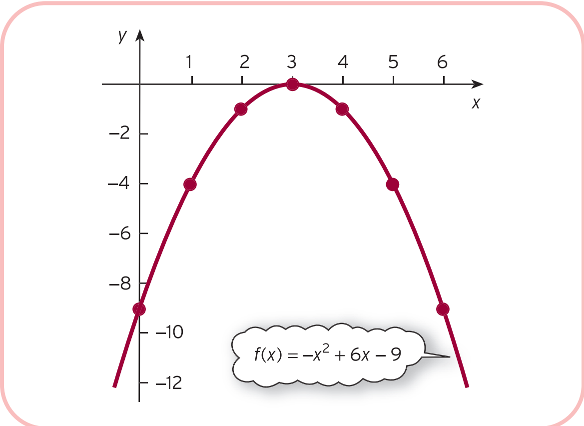


Figure S2.2

(c)

x	−2	−1	0	1	2	3	4
$f(x)$	−22	−12	−6	−4	−6	−12	−22

The graph is sketched in Figure S2.3.

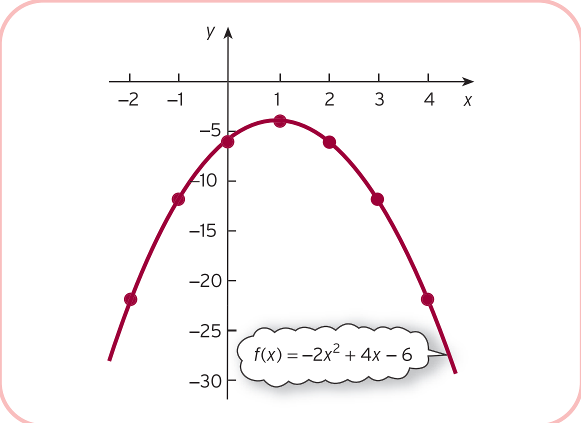


Figure S2.3

5 (a) Step 1

The coefficient of x^2 is 2, which is positive, so the graph is U-shaped.

Step 2

The constant term is −6, so the graph crosses the vertical axis at $y = -6$.

Step 3

The quadratic equation

$$2x^2 - 11x - 6 = 0$$

has solution

$$x = \frac{-(-11) \pm \sqrt{((-11)^2 + 4(2)(6))}}{2(2)}$$

$$= \frac{11 \pm \sqrt{(121 + 48)}}{4}$$

$$= \frac{11 \pm \sqrt{169}}{4}$$

$$= \frac{11 \pm 13}{4}$$

so the graph crosses the horizontal axis at $x = -\frac{1}{2}$ and $x = 6$.

In fact, we can use symmetry to locate the coordinates of the turning point on the curve. The x coordinate of the minimum occurs halfway between $x = -\frac{1}{2}$ and $x = 6$ at

$$x = \frac{1}{2}\left(-\frac{1}{2} + 6\right) = \frac{11}{4}$$

The corresponding y coordinate is

$$2\left(\frac{11}{4}\right)^2 - 11\left(\frac{11}{4}\right) - 6 = -\frac{169}{8}$$

The graph is sketched in Figure S2.4.

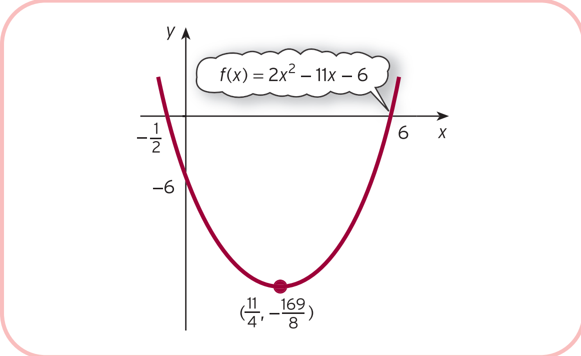


Figure S2.4

(b) Step 1

The coefficient of x is 1, which is positive, so the graph is U-shaped.

Step 2

The constant term is 9, so the graph crosses the vertical axis at $y = 9$.

Step 3

The quadratic equation

$$x^2 - 6x + 9 = 0$$

has solution

$$x = \frac{-(-6) \pm \sqrt{((-6)^2 - 4(1)(9))}}{2(1)}$$

$$= \frac{6 \pm \sqrt{(36 - 36)}}{2}$$

$$= \frac{6 \pm \sqrt{0}}{2} = 3$$

so the graph crosses the x axis at $x = 3$.

The graph is sketched in Figure S2.5.

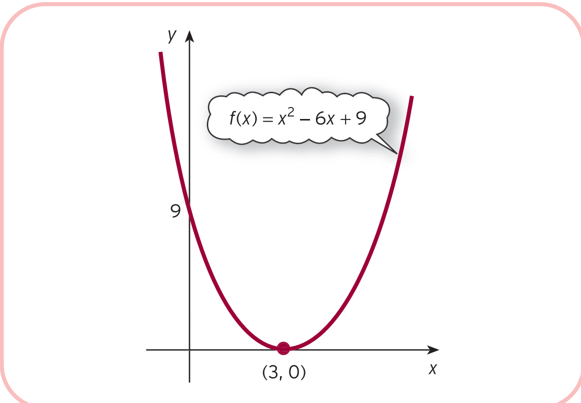


Figure S2.5

6 **(a)** Figure S2.4 shows that the graph is on or below the x-axis for values of x between $-\frac{1}{2}$ and 6 with the function taking the value 0 at the end-points so the solution is $-\frac{1}{2} \le x \le 6$.

(b) Figure S2.5 shows that the graph is always on or above the x-axis taking the value 0 at the one point, $x = 3$. However, because the inequality is strict we need to exclude this point so the solution consists of all values of x except for $x = 3$.

7 In equilibrium, $Q_S = Q_D = Q$, so the supply and demand equations become

$$P = 2Q^2 + 10Q + 10$$

$$P = Q^2 - 5Q + 52$$

Hence

$$2Q^2 + 10Q + 10 = -Q^2 - 5Q + 52$$

$$3Q^2 + 15Q - 42 = 0$$

(collecting like terms)

$$Q^2 + 5Q - 14 = 0$$

(dividing both sides by 3)

$$Q = \frac{-5 \pm \sqrt{((5)^2 - 4(1)(-14))}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{81}}{2}$$

$$= \frac{-5 \pm 9}{2}$$

so $Q = -7$ and $Q = 2$. Ignoring the negative solution gives $Q = 2$. From the supply equation, the corresponding equilibrium price is

$$P = 2(2)^2 + 10(2) + 10 = 38$$

As a check, the demand equation gives

$$P = -(2)^2 - 5(2) + 52 = 38$$

Exercise 2.1 (p. 143)

1 **(a)** ± 9 **(b)** ± 6 **(c)** ± 2 **(d)** $-2, 4$ **(e)** $-9, -1$

2 **(a)** $1, -3$ **(b)** $^1/_2, -10$ **(c)** $0, -5$ **(d)** $-^5/_3, {}^9/_4$ **(e)** $^5/_4, 5$

3 **(a)** 0.44, 4.56 **(b)** −2.28, 0.22 **(c)** −0.26, 2.59 **(d)** −0.30, 3.30 **(e)** −2 **(f)** no solution

4 **(a)** −4, 4 **(b)** 0, 100 **(c)** 5, 17 **(d)** 9 **(e)** no solution

5 The graphs are sketched in Figure S2.6.

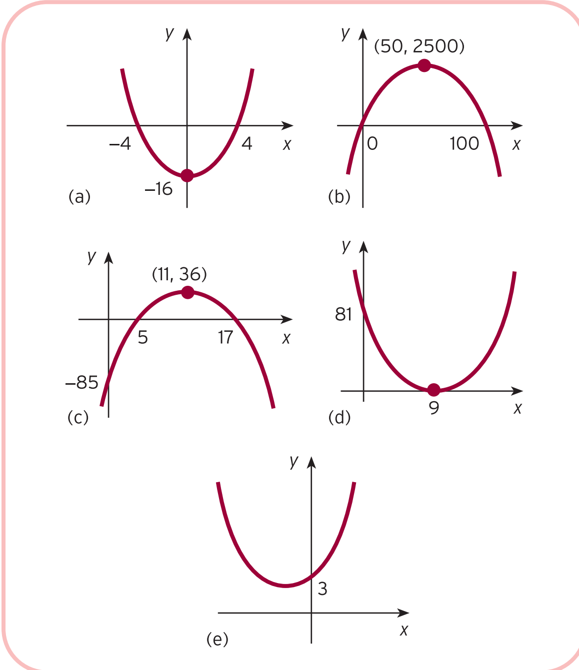


Figure S2.6

6 **(a)** $x \le -4, x \ge 4$ **(b)** $0 < x < 100$ **(c)** $5 \le x \le 17$ **(d)** $x = 9$ **(e)** all values of x

7 $P = 36, Q = 4$

8 $P = 22, Q = 3$

Exercise 2.1* (p. 144)

1 **(a)** ± 13 **(b)** $-3, 13$ **(c)** $-2, 9$

2 $-7d, d$

3 **(a)** $3, -8$ **(b)** $\frac{2}{3}, -\frac{9}{2}$ **(c)** $0, \frac{3}{4}$ **(d)** $\frac{1}{6}$ (twice) **(e)** $2, -1, 4$

4 **(a)** 7, 8 **(b)** 0.22, 2.28 **(c)** ± 3 **(d)** 7 (twice) **(e)** no solutions **(f)** 10, 19

5 **(a)** $x \le -8, x \ge 8$ **(b)** $1 \le x \le 9$ **(c)** $-7 < x < -\frac{1}{2}$ **(d)** $-1 \le x \le \frac{5}{3}$ **(e)** $x = -1$

6 $c = 12$; 6

7 $k = 27$

8 120.76

9 **(a)** $P = 18$ **(b)** $B = 15$

10 $P = 5, Q = 65$

Section 2.2

Practice Problems

1 $TR = PQ = (1000 - Q)Q = 1000Q - Q^2$

Step 1

The coefficient of Q^2 is negative, so the graph has an inverted U shape.

Step 2

The constant term is zero, so the graph crosses the vertical axis at the origin.

Step 3

From the factorization

$$TR = (1000 - Q)Q$$

the graph crosses the horizontal axis at $Q = 0$ and $Q = 1000$.

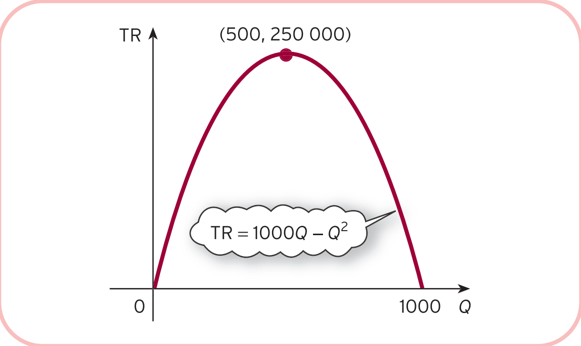


Figure S2.7

The graph is sketched in Figure S2.7. By symmetry the parabola reaches its maximum halfway between 0 and 1000 at $Q = 500$. The corresponding value of TR is

$$TR = 1000(500) - (500)2 = 250\ 000$$

From the demand equation, when $Q = 500$,

$$P = 1000 - 500 = 500$$

2 $TC = 100 + 2Q$

$$AC = \frac{100 + 2Q}{Q} = \frac{100}{Q} + 2$$

The graph of the total cost function is sketched in Figure S2.8.

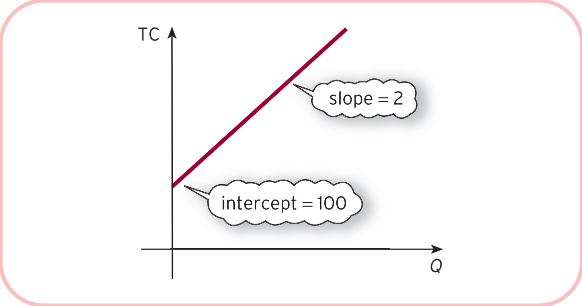


Figure S2.8

One possible table of function values for the average cost function is

Q	10	25	50	100	200
AC	12	6	4	3	2.5

The graph of the average cost function is sketched in Figure S2.9.

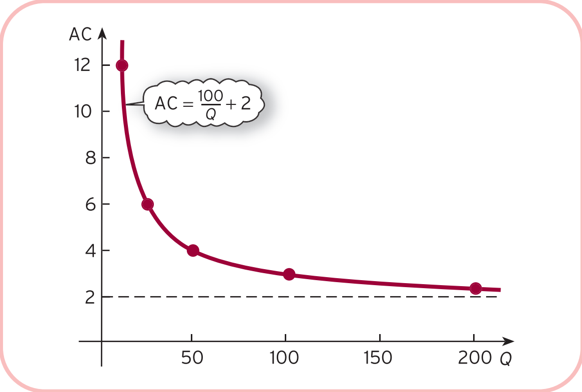


Figure S2.9

In fact, it is not necessary to plot the tabulated values if all that is required is a rough sketch. It is obvious that if a very small number is put into the AC function then a very large number is produced because of the term $100/Q$. For example, when $Q = 0.1$

$$AC = \frac{100}{0.1} + 2 = 1002$$

It should also be apparent that if a very large number is put into the average cost function then the term $100/Q$ is insignificant, so AC is approximately 2. For example, when $Q = 10\ 000$

$$AC = \frac{100}{10\ 000} + 2 = 2.01$$

The graph of AC therefore 'blows up' near $Q = 0$ but settles down to a value just greater than 2 for large Q. Consequently, the general shape of the graph shown in Figure S2.9 is to be expected.

3 $TC = 25 + 2Q$

$TR = PQ = (20 - Q)Q = 20Q - Q^2$

Hence

$$\pi = TR - TC$$
$$= (20Q - Q^2) - (25 + 2Q)$$
$$= 20Q - Q^2 - 25 - 2Q$$
$$= -Q^2 + 18Q - 25$$

Step 1

The coefficient of Q^2 is negative, so the graph has an inverted U shape.

Step 2

The constant term is −25, so the graph crosses the vertical axis at −25.

Step 3

The quadratic equation

$$-Q^2 + 18Q - 25 = 0$$

has solutions

$$Q = \frac{-18 \pm \sqrt{(324 - 100)}}{-2} = \frac{-18 \pm 14.97}{-2}$$

so the graph crosses the horizontal axis at $Q = 1.52$ and $Q = 16.48$.

The graph of the profit function is sketched in Figure S2.10.

(a) If $\pi = 31$ then we need to solve

$$-Q^2 + 18Q - 25 = 31$$

that is,

$$-Q^2 + 18Q - 56 = 0$$

$$Q = \frac{-18 \pm \sqrt{(324 - 224)}}{-2} = \frac{18 \pm 10}{-2}$$

so $Q = 4$ and $Q = 14$.

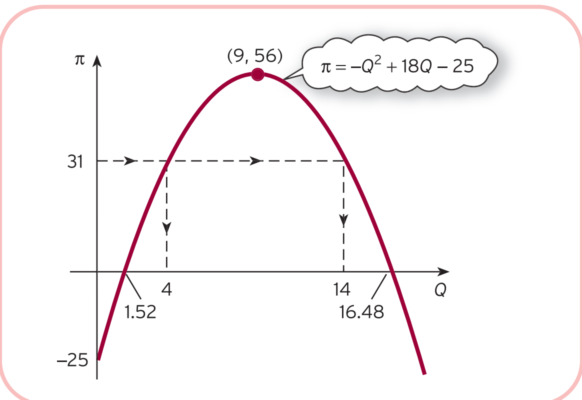


Figure S2.10

These values can also be found by drawing a horizontal line $\pi = 31$ and then reading off the corresponding values of Q from the horizontal axis as shown on Figure S2.10.

(b) By symmetry the parabola reaches its maximum halfway between 1.52 and 16.48: that is, at

$$Q = \tfrac{1}{2}(1.52 + 16.48) = 9$$

The corresponding profit is given by

$$\pi = -(9)^2 + 18(9) - 25 = 56$$

Exercise 2.2 (p. 156)

1 **(a)** $P = 50$; $TR = 500$ **(b)** $TC = 150$ **(c)** $\pi = 350$

2 **(a)** $4Q$ **(b)** 7 **(c)** $10Q - 4Q^2$

The graphs are sketched in Figures S2.11, S2.12 and S2.13.

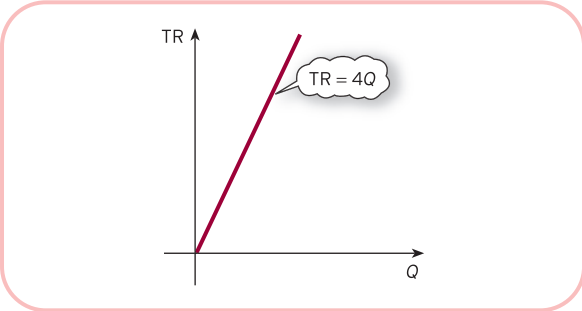


Figure S2.11

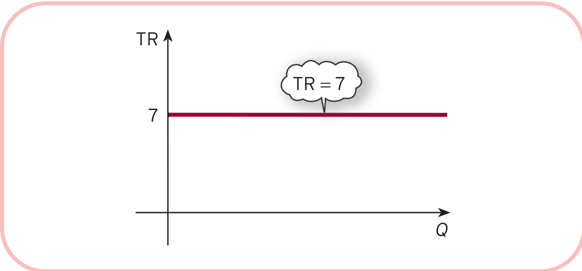


Figure S2.12

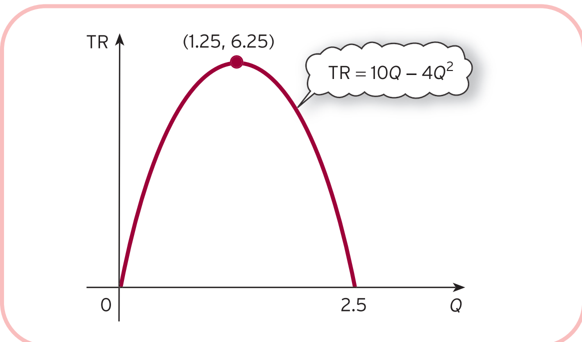


Figure S2.13

3 (a) $P = 50 - 4Q$

(b) $P = \dfrac{10}{Q}$

4 $\text{TC} = 500 + 10Q \quad \text{AC} = \dfrac{500}{Q} + 10$

The graphs are sketched in Figures S2.14 and S2.15.

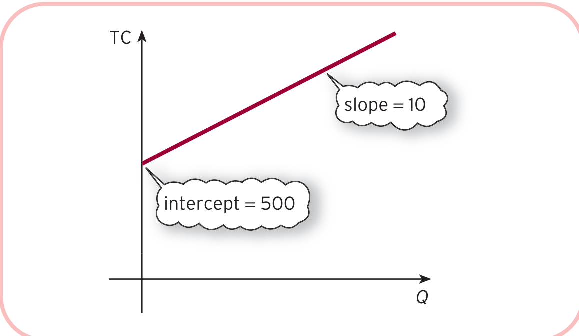


Figure S2.14

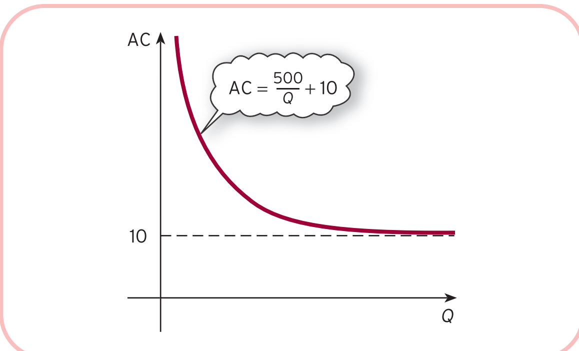


Figure S2.15

5 $\text{TC} = Q^2 + Q + 1;\ \text{AC} = Q + 1 + \dfrac{1}{Q}$

The graphs are sketched in Figures S2.16 and S2.17.

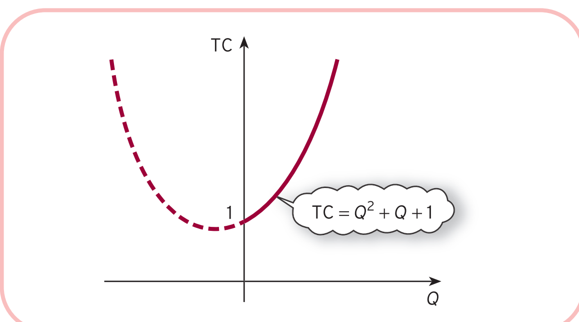


Figure S2.16

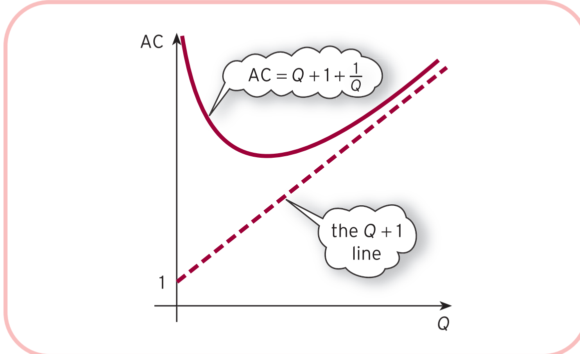


Figure S2.17

6 $\pi = -2Q^2 + 20Q - 32$ (a) 2, 8 (b) 20 (c) 5

7 The graphs of TR and TC are sketched in Figure S2.18.

(a) 1, 5 (b) 3

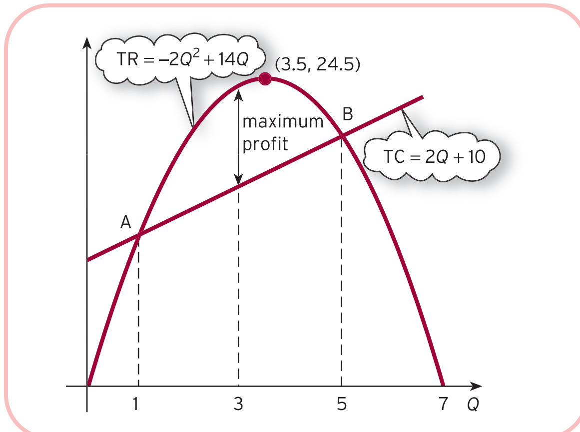


Figure S2.18

8 (a) $\text{TR} = PQ = 60Q - Q^2$

The graph is an inverted parabola passing through (0, 0), (60, 0).

(b) $TC = 100 + (Q + 6)Q$

$= Q^2 + 6Q + 100$

Divide by Q to find AC.

31, 26, $27\frac{2}{3}$, 31; $Q = 10$

(c) $\pi = \text{TR} - \text{TC}$

$= 60Q - Q^2 - (Q^2 + 6Q + 100)$

$= -2Q^2 + 54Q - 100$

e.g. multiply out brackets

$Q = 2, 25$

When $Q = 13.5$, $\pi = 264.5$

Exercise 2.2* (p. 157)

1 15 and $\frac{2}{3}$; max profit is $154\frac{1}{12}$ which occurs at $Q = 7\frac{5}{6}$

2 $a + b + c = 9$

$4a + 2b + c = 34$

$9a + 3b + c = 19$

$a = -20$, $b = 85$, $c = -56$; $\pi = -36$

3 **(a)** 20 **(b)** 800

4 **(a)** $\text{TR} = \frac{cQ - bQ^2}{a}$ **(b)** $\text{TC} = eQ + d$

(c) $\text{AC} = e + \frac{d}{Q}$ **(d)** $\pi = \frac{-bQ^2 + (c - ae)Q - ad}{a}$

5 **(a)** Ennerdale: $0.5aN + 0.25(1 - a)N$

North Borsetshire: $15 + 0.3N$

Equate two expressions to get result.

L-shaped curve, tending to 1/5

(b) $a < 0.4$ leads to $N > 300$, so choose Ennerdale if number of withdrawals exceeds 300.

6 1000, 1004.08, . . . , 1750; 22.8.

7 **(a)** 5, 30 **(b)** 18

Section 2.3

Practice Problems

1 **(a)** 100 **(b)** 10 **(c)** 1 **(d)** 1/10

(e) 1/100 **(f)** 1 **(g)** −1 **(h)** 1/343

(i) 81 **(j)** 72 101 **(k)** 1

2 **(a)** 4, because $4^2 = 16$.

(b) 3, because $3^3 = 27$.

(c) 32, because $4^{5/2} = (4^{1/2})^5 = 2^5$.

(d) ¼, because $8^{-2/3} = (8^{1/3})^{-2} = 2^{-2} = 1/2^2$.

(e) 1, because $1^n = 1$ for any index, n.

3 **(a)** $(x^{3/4})^8 = x^{(3/4)\times 8} = x^6$ (rule 3)

(b) $x^2 \div x^{3/2} = x^{2-(3/2)} = x^{1/2}$ (rule 2)

(c) $(x^2y^4)^3 = (x^2)^3(y^4)^3$ (rule 4)

$= x^{2\times 3}y^{4\times 3}$ (rule 3)

$= x^6y^{12}$

(d) $\sqrt{x}(x^{5/2} + y^3) = x^{1/2}(x^{5/2} + y^3)$

(definition of $b^{1/n}$)

$= x^{1/2}x^{5/2} + x^{1/2}y^3$

(multiply out the brackets)

$= x^{(1/2)+(5/2)} + x^{1/2}y^3$ (rule 1)

$= x^3 + x^{1/2}y^3$

The term $x^{1/2}y^3$ cannot be simplified, because $x^{1/2}$ and y^3 have different bases.

4 **(a)** $f(K, L) = 7KL^2$

$f(\lambda K, \lambda L) = 7(\lambda K)(\lambda L)^2$

$= 7\lambda K\, \lambda^2 L^2$ (rule 4)

$= (\lambda\lambda^2)(7KL^2)$

$= \lambda^3 f(K, L)$ (rule 1)

Increasing returns to scale because 3 > 1.

(b) $f(K, L) = 50K^{1/4}L^{3/4}$

$f(\lambda K, \lambda L) = 50(\lambda K)^{1/4}(\lambda L)^{3/4}$

$= 50\lambda^{1/4}K^{1/4}\lambda^{3/4}L^{3/4}$ (rule 4)

$= (\lambda^{1/4}\lambda^{3/4})(50K^{1/4}L^{3/4})$

$= \lambda^1 f(K, L)$ (rule 1)

Constant returns to scale.

5 **(1)** **(a)** 3 **(b)** 2 **(c)** 1 **(d)** 0 **(e)** −1 **(f)** −2

(2) Same as part (1), because if $M = 10^n$ then $\log_{10}M = n$.

(3) On most calculators there are two logarithm function keys, $\log_{10}$ (possibly labelled log or $\log_{10}$) and ln (possibly labelled ln or $\log_e$). The latter is known as the natural logarithm and we introduce this function in the next section. This question wants you to evaluate logarithms to base 10, so we use the key [log].

Warning: there is no standard layout for the keyboard of a calculator. It may be necessary for you first to use the shift key (sometimes called the inverse function or second function key) to activate the $\log_{10}$ function.

6 **(a)** $\log_b\left(\frac{x}{y}\right) + \log_b z$ (rule 2)

$= \log_b\left(\frac{xz}{y}\right)$ (rule 1)

(b) $\log_b x^4 + \log_b y^2$ (rule 3)

$= \log_b(x^4y^2)$ (rule 1)

7 **(a)** $3^x = 7$

$\log(3^x) = \log 7$

(take logarithms of both sides)

$x \log 3 = \log 7$ (rule 3)

$x = \frac{\log 7}{\log 3}$

(divide both sides by log 3)

$x = \frac{0.845\ 098\ 040}{0.477\ 121\ 255}$

(using base 10 on a calculator)

$x = 1.77$

(to two decimal places)

(b) $5(2)^x = 10^x$

$$\log[5(2)^x] = \log(10)^x$$

(take logarithms of both sides)

$$\log 5 + \log(2^x) = \log(10)^x \quad \text{(rule 1)}$$

$$\log 5 + x \log 2 = x \log 10 \quad \text{(rule 3)}$$

$$x(\log 10 - \log 2) = \log 5$$

(collect terms and factorize)

$$x \log 5 = \log 5 \quad \text{(rule 2)}$$

$$x = 1$$

(divide both sides by log 5)

This is, of course, the obvious solution to the original equation! Did you manage to spot this for yourself *before* you started taking logs?

Exercise 2.3 (p. 177)

1 **(a)** 64 **(b)** 2 **(c)** 1/3 **(d)** 1 **(e)** 1 **(f)** 6 **(g)** 4 **(h)** 1/343

2 **(a)** a^{11} **(b)** b^5 **(c)** c^6 **(d)** x^2y^2 **(e)** x^3y^6 **(f)** y^{-4} **(g)** x^4 **(h)** f^7 **(i)** y^3 **(j)** x^5

3 **(a)** $x^{1/2}$ **(b)** x^{-2} **(c)** $x^{1/3}$ **(d)** x^{-1} **(e)** $x^{-1/2}$ **(f)** $x^{3/2}$

4 **(a)** 3600 **(b)** 200 000

5 The functions in parts **(a)** and **(b)** are homogeneous of degree 7/12 and 2 respectively, so **(a)** displays decreasing returns to scale and **(b)** displays increasing returns to scale. The function in part **(c)** is not homogeneous.

6 **(a)** 2 **(b)** −1 **(c)** −3 **(d)** 6 **(e)** ½ **(f)** 1

7 **(a)** 2 **(b)** 1 **(c)** 0 **(d)** ½ **(e)** −1

8 **(a)** $\log_b(xz)$ **(b)** $\log_b\left(\frac{x^3}{y^2}\right)$ **(c)** $\log_b\left(\frac{y}{z^3}\right)$

9 **(a)** $2\log_b x + \log_b y$

(b) $\log_b x - 2\log_b y$

(c) $2\log_b x + 7\log_b y$

10 **(a)** 1.29 **(b)** 1.70 **(c)** 6.03 **(d)** 8.31

11 **(1)** **(a)** 5 **(b)** $\left(-\frac{1}{2}\right)$

(2) $\log_b\left(\frac{x^2}{y^4}\right)$

(3) 69.7

12 **(1)** **(a)** 4 **(b)** −2 **(c)** 2

(2) **(a)** x^3y **(b)** $x^{15}y^5$ **(c)** x^2y^2

Exercise 2.3* (p. 178)

1 **(a)** 8 **(b)** 1/32 **(c)** 625 **(d)** $2\frac{1}{4}$ **(e)** 2/3

2 **(a)** y^2 **(b)** xy^2 **(c)** x^4y^2 **(d)** 1 **(e)** 2 **(f)** $5pq^2$

3 **(a)** x^{-7} **(b)** $x^{1/4}$ **(c)** $x^{-3/2}$ **(d)** $2x^{11/2}$ **(e)** $8x^{-4/3}$

4 $3x^3y^7$

5 $A[b(\lambda K)^\alpha + (1-b)(\lambda L)^\alpha]^{1/\alpha}$

$$= A[b\lambda^\alpha K^\alpha + (1-b)\lambda^\alpha L^\alpha]^{1/\alpha} \quad \text{(rule 4)}$$

$$= A[(\lambda^\alpha)(bK^\alpha + (1-b)L^\alpha)]^{1/\alpha} \quad \text{(factorize)}$$

$$= A(\lambda^\alpha)^{1/\alpha}[bK^\alpha + (1-b)L^\alpha]^{1/\alpha} \quad \text{(rule 4)}$$

$$= \lambda A[bK^\alpha + (1-b)L^\alpha]^{1/\alpha} \quad \text{(rule 3)}$$

So $f(\lambda K, \lambda L) = \lambda^1 f(K, L)$ as required. This is known as the constant elasticity of substitution (CES) production function.

6 **(a)** 2/3 **(b)** 3 **(c)** ¼

7 **(a)** $\log_b(1)$ **(b)** $\log_b\left(\frac{x^3}{y^2}\right)$ **(c)** $\log_b\left(\frac{x^5y}{z^2}\right)$ **(d)** $\log_b(b^2x^3)$

8 **(a)** $2\log_b x + 3\log_b y + 4\log_b z$

(b) $4\log_b x - 2\log_b y - 5\log_b z$

(c) $\log_b x - \frac{1}{2}\log_b y - \frac{1}{2}\log_b z$

9 **(a)** $-q$ **(b)** $2p + q$ **(c)** $q - 4r$ **(d)** $p + q + 2r$

10 **(a)** 78.31 **(b)** 1.48 **(c)** 3 **(d)** 0.23

11 **(a)** $x \le 0.386$ (3 dp)

(b) $x > 14.425$ (Notice that the inequality is > here.)

12 $x = 3$ (Note that the second solution of your quadratic, $x = -5$, is not valid.)

13 **(2)** $\frac{2}{3}$; constant returns to scale

14 **(1)** **(a)** $\frac{2}{3}$ **(b)** $-\frac{1}{2}$

(2) $y = \frac{1}{7}x^2$

15 $\log_{10}x^2 - \log_{10}\sqrt{y} - \log_{10}10$

$$= \log_{10}\left(\frac{x^2}{\sqrt{y}}\right) - 1$$

$$= \log_{10}\left(\sqrt{\frac{x^4}{y}}\right) - 1$$

Section 2.4

Practice Problems

1

x	−3	−2	−1	0	1	2	3
3^x	0.04	0.11	0.33	1	3	9	27
3^{-x}	27	9	3	1	0.33	0.11	0.04

The graphs of 3^x and 3^{-x} are sketched in Figures S2.19 and S2.20, respectively.

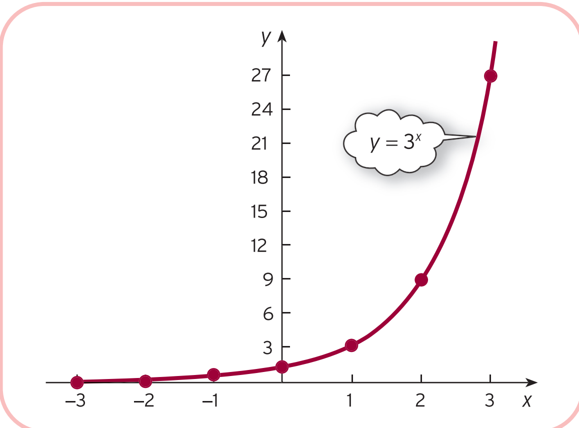


Figure S2.19

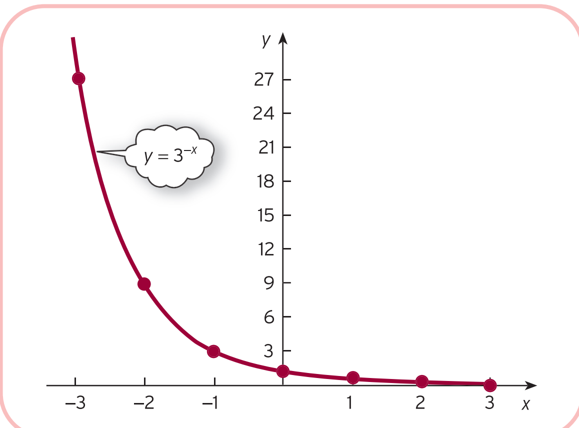


Figure S2.20

2 **(a)** 2.718 145 927, 2.718 268 237, 2.718 280 469.

(b) 2.718 281 828; values in part **(a)** are getting closer to that of part **(b)**.

3 **(1)** Substituting $t = 0$, 10, 20 and 30 gives

(a) $y(0) = \dfrac{55}{1 + 800e^{0}} = 0.07\%$

(b) $y(10) = \dfrac{55}{1 + 800e^{-3}} = 1.35\%$

(c) $y(20) = \dfrac{55}{1 + 800e^{-6}} = 18.44\%$

(d) $y(30) = \dfrac{55}{1 + 800e^{-9}} = 50.06\%$

(2) As t increases, $e^{-0.3t}$ goes to zero, so y approaches

$$\frac{55}{1 + 800(0)} = 55\%$$

(3) A graph of y against t, based on the information obtained in parts (1) and (2), is sketched in Figure S2.21. This shows that, after a slow start, camcorder ownership grows rapidly between $t = 10$ and 30. However, the rate of growth then decreases as the market approaches its saturation level of 55%.

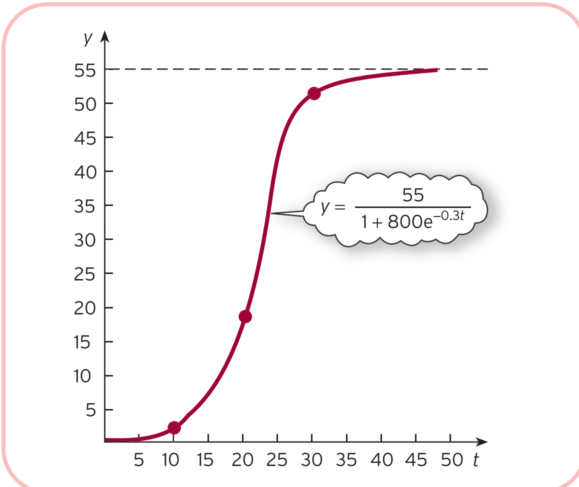


Figure S2.21

4 **(a)** $\ln a^2 + \ln b^3$ (rule 1)

$= 2 \ln a + 3 \ln b$ (rule 3)

(b) $\ln x^{1/2} - \ln y^3$ (rule 3)

$= \ln \left(\dfrac{x^{1/2}}{y^3}\right)$ (rule 2)

5 **(a)** Putting $t = 0$ and 2 into the expression for TR gives

$TR = 5e^{0} = \$5$ million

$TR = 5e^{-0.3} = \$3.7$ million

(b) To solve $5e^{-0.15t} = 2.7$ we divide by 5 to get $e^{-0.15t} = 0.54$ and then take natural logarithms, which gives

$-0.15t = \ln(0.54) = -0.62$

Hence $t = 4$ years.

6 **(1)** Missing numbers are 0.99 and 2.80.

(2) The graph is sketched in Figure S2.22.

Intercept, 0.41; slope, 0.20.

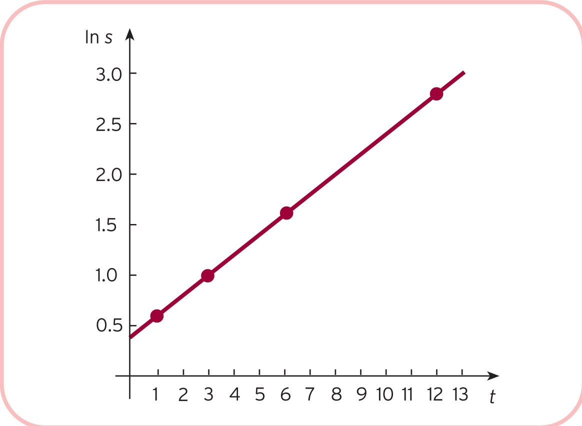


Figure S2.22

(3) $A = 0.2$, $B = e^{0.41} = 1.5$

(4) (a) 9100 (b) 2.4×10^8; answer to part (b) is unreliable since $t = 60$ is well outside the range of given data.

Exercise 2.4 (p. 190)

1 (1) (a) 33 (b) 55 (c) 98

(2) 100

(3) The graph of N against t is sketched in Figure S2.23.

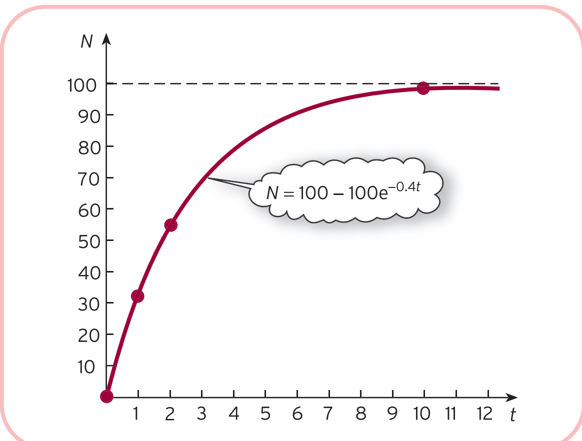


Figure S2.23

The graph sketched in Figure S2.23 is called a learning curve. It shows that immediately after training the worker can produce only a small number of items. However, with practice, output quickly increases and finally settles down at a daily rate of 100 items.

2 (a) $\ln x + \ln y$ (b) $\ln x + 4 \ln y$

(c) $2 \ln x + 2 \ln y$ (d) $5 \ln x - 7 \ln y$

(e) $\frac{1}{2} \ln x - \frac{1}{2} \ln y$ (f) $\frac{1}{2} \ln x + \frac{3}{2} \ln y - \frac{1}{2} \ln z$

3 (a) $\ln x^3$ (b) $\ln\left(\frac{x^4 z^5}{x^3}\right)$

4 (a) 1.77 (b) −0.80 (c) no solution

(d) 0.87 (e) 0.22 (f) 0.35

5 $A = 50\,000$, $a = 0.137$ (a) \$25 205 (b) \$0

6 All five graphs are increasing throughout.

Graphs of x^2, x^3 and e^x bend upwards, i.e. the slope increases with increasing x, whereas x has a constant slope and $\sqrt{x}$ has a decreasing slope.

With the *exception* of e^x, all graphs pass through (0, 0) and (1, 1). On $0 < x < 1$, $x^3 < x^2 < x < \sqrt{x}$ whereas on $x > 1$ the order is reversed.

Exercise 2.4* (p. 191)

1 \$162.19

2 (a) 1.13 (b) 1.79 (c) 8.77

3 The equation

$$N = c(1 - e^{-kt}) = c - ce^{-kt}$$

rearranges as

$$e^{-kt} = \frac{c - N}{c}$$

Taking logarithms gives

$$-kt = \ln\left(\frac{c - N}{c}\right)$$

$$t = -\frac{1}{k}\ln\left(\frac{c - N}{c}\right)$$

$$= \frac{1}{k}\ln\left(\frac{c - N}{c}\right)^{-1} = \frac{1}{k}\ln\left(\frac{c}{c - N}\right)$$

(a) 350 000

(b) 60 days.

(c) Market saturation level is 700 000, which is less than the three-quarters of a million copies needed to make a profit, so the proprietor should sell.

4 $\ln Q = \ln 3 + \frac{1}{2}\ln L + \frac{1}{3}\ln K$

Putting $y = \ln Q$, $x = \ln K$ gives

$$y = \tfrac{1}{3}x + (\ln 3 + \tfrac{1}{2}\ln L)$$

which is of the form '$y = ax + b$'.

Slope = $\frac{1}{3}$, intercept = $\ln 3 + \frac{1}{2}\ln L$.

5 (a) $\ln Q = \ln(AL^n) = \ln A + \ln L^n = \ln A + n \ln L$

(b)

$\ln L$	0	0.69	1.10	1.39	1.61
$\ln Q$	−0.69	−0.46	−0.33	−0.22	−0.16

(c) $n = 0.34$, $A = 0.50$

6 (a) $3y^2 + 13y - 10$

(b) Put $y = e^x$ in part (a) to deduce −0.405.

7 (a) $y = \frac{1}{b}\ln\left(\frac{x}{a}\right)$ (b) $x = \frac{1}{2}\ln(e^y - 3)$

8 All five graphs have the same basic shape and pass through (1, 0).

On $0.2 < x < 1$, $\ln x < \log_6 x < \log_{10} x$, whereas on $x > 1$ the order is reversed.

9 (a) $P - 100 = -\frac{2}{3}Q^n$ (subtract 100 from both sides)

$-\frac{3}{2}(P - 100) = Q^n$ (divide both sides by $-\frac{2}{3}$)

$150 - 1.5P = Q^n$ (multiply out the brackets)

$\ln(150 - 1.5P) = \ln(Q^n) = n \ln Q$

(take logs and use rule 3)

which is of the form $y = ax + b$ with $a = n$ and $b = 0$.

(b)

$\ln Q$	2.30	3.91	4.09
$\ln(150 - 1.5P)$	2.01	3.11	3.40

$\ln Q$	4.61	5.29	5.99
$\ln(150 - 1.5P)$	3.81	4.32	4.79

$n = 0.8$

Chapter 3

Section 3.1

Practice Problems

1 (a) $\frac{10}{100} \times 2.90 = 0.1 \times 2.90 = \$0.29$

(b) $\frac{75}{100} \times 1250 = 0.75 \times 1250 = \$937.50$

(c) $\frac{24}{100} \times 580 = 0.24 \times 580 = \$139.20$

2 (a) The rise in annual sales is

$55\,000 - 50\,000 = 5000$

As a fraction of the original this is

$$\frac{5000}{500\,000} = \frac{10}{100}$$

so the percentage rise is 10%.

(b) As a fraction

15% is the same as $\frac{15}{100} = 0.15$

so the tax is

$0.15 \times 1360 = 204$

Hence the consumer pays

$1360 + 204 = \$1564$

(c) As a fraction

7% is the same as $\frac{7}{100} = 0.07$

so the fall in value is

$0.07 \times 9500 = 665$

Hence the final value is

$9500 - 665 = \$8835$

3 (a) The scale factor is

$$1 + \frac{13}{100} = 1.13$$

We are going forwards in time, so we *multiply* to get

$6.5 \times 1.13 = \$7.345$ million

(b) The scale factor is

$$1 + \frac{63}{100} = 1.63$$

We are going backwards in time, so we *divide* to get

$1.24 \div 1.63 = \$76$ billion
(correct to the nearest billion)

(c) The scale factor is

$$\frac{123\,050}{115\,000} = 1.07$$

which can be thought of as

$$1 + \frac{7}{100}$$

so the rise is 7%.

4 (a) The scale factor is

$$1 - \frac{65}{100} = 0.35$$

so the new level of output is

$25\,000 \times 0.35 = 8750$

(b) The scale factor is

$$1 - \frac{24}{100} = 0.76$$

so before restructuring the number of employees was

$570 \div 0.76 = 750$

(c) The scale factor is

$$\frac{2.10}{10.50} = 0.2$$

which can be thought of as

$$1 - \frac{80}{100}$$

so the fall is 80% (not 20%!).

5 (a) $1.3 \times 1.4 = 1.82$, which corresponds to an 82% increase.

(b) $0.7 \times 0.6 = 0.42$, which corresponds to a 58% decrease.

(c) $1.1 \times 0.5 = 0.55$, which corresponds to a 45% decrease.

6 100, 101.5, 105.4, 104.3, 106.9

7 **(a)** 5.7% increase

(b) $\frac{105.7}{89.3} = 1.184$ so 18.4% increase

(c) $\frac{100}{89.3} = 1.120$ so 12.0% increase

8 The 2000 adjusted salary is

$17.3 \times 1.049 = 18.1$

The 2002 adjusted salary is

$\frac{19.8}{1.043} = 19.0$

and so on. The complete set of 'constant 2001 prices' is listed in Table S3.1.

Table S3.1

	Year				
	00	01	02	03	04
Real salaries	18.1	18.1	19.0	21.7	23.2

During 2000/1 salaries remain unchanged in real terms. However, since 2001 salaries have outpaced inflation with steady increases in real terms.

Exercise 3.1 (p. 209)

1 **(a)** $\frac{7}{20}$ **(b)** $\frac{22}{25}$ **(c)** $2\frac{1}{2}$ **(d)** $\frac{7}{40}$ **(e)** $\frac{1}{500}$

2 **(a)** 1.2 **(b)** 7.04 **(c)** 2190.24 **(d)** 62.72

3 **(a)** 1.19 **(b)** 3.5 **(c)** 0.98 **(d)** 0.57

4 **(a)** 4% increase **(b)** 42% increase

(c) 14% decrease **(d)** 245% increase

(e) 0.25% increase **(f)** 96% decrease

5 **(a)** \$18.20 **(b)** 119 244 **(c)** \$101.09

(d) \$1610 **(e)** \$15 640

6 **(a)** \$15.50 **(b)** \$10.54 **(c)** 32%

7 \$864

8 **(a)** \$26 100 **(b)** 31% (nearest percentage)

9 **(a)** 37.5% increase **(b)** 8.9% increase

(c) 6.25% decrease

10 \$11.6 million

11 **(1)** 1985

(2) **(a)** 30% **(b)** 52.3% **(c)** 13.1% **(d)** 9.4%

(3) **(a)** 25% **(b)** 44% **(c)** 10.6% **(d)** 11.1%

(4) Public transport costs have risen at a faster rate than private transport throughout the period 1985–2000. However, for the final 5 years there are signs that the trend has stopped and has possibly reversed.

12 964, 100, 179, 750; e.g. seasonal variations

13 **(a)** 83.3, 100, 91.7, 116.7, 125, 147.9

(b) $48 \times 1.35 = 64.8$

(c) $\frac{44}{x} \times 100 = 73 \quad \Rightarrow \quad x = \frac{44 \times 100}{73} = 60.3$

So, the year is 2003.

Exercise 3.1* (p. 211)

1 \$977.50

2 **(a)** \$90

(b) 40%; the 20% discount is applied to the price after the first reduction not the original.

3 **(a)** \$720

(b) 40% of the new price is less than 40% of the original.

(c) divide by 0.6

4 **(a)** \$850 **(b)** 19% decrease

(c) 23.5% increase

5 **(a)** 100, 101.7, 113.1, 116.9

(b) Real output: 236, 229.2, 244.7, 244.7

Index: 100, 97.1, 103.7, 103.7

(c) In real terms, spending on education fell by 2.9% in 1995, increased by 6.8% in 1996 and remained unchanged in 1997.

6 **(a)** 1 and 6, respectively.

(b) 142, 150 **(c)** 94, 87, 83, 75, 79

(d) 1.1 million and 1.6 million.

7 **(a)** 239.2, 242, 243.69, 243.43, 250.73, 258.56
98.8, 100, 100.7, 100.6, 103.6, 106.8

(b) 297 **(c)** 3.5%

8 108.4, 119.5. These values reflect the rises given to the bulk of employees who fall into categories B and C. The generous rises given to senior management have had little effect on the index because there are only 7 (out of 390) employees in category D.

9 111.7, 173.6. These indices are higher than before. Although the total number of employees has remained almost unchanged, many of these have been promoted to the senior management team, thereby increasing the total wage bill.

Paasche index uses up-to-date information whereas Laspeyre uses only quantities relating to the base year, which become more irrelevant over time.

Laspeyre index is easier to calculate and interpret. Also we can compare two or more Laspeyre indices since they relate to the same basket of goods. It may be impossible to calculate the Paasche index since data about current performance may not be readily available at the time.

Section 3.2

Practice Problems

1 The calculations are summarized in Table S3.2.

Table S3.2

End of year	Interest ($)	Investment ($)
1	80	1080
2	86.40	1166.40
3	93.31	1259.71
4	100.78	1360.49
5	108.84	1469.33
6	117.55	1586.88
7	126.95	1713.83
8	137.11	1850.94
9	148.08	1999.02
10	159.92	2158.94

2 $S = 1000(1.08)^{10} = \$2158.92$

The slight discrepancy between the two answers obtained in Practice Problems 1 and 2 arises because the intermediate results in Practice Problem 1 are rounded to 2 decimal places.

3

$$9000(1.03)^n = 10\,000$$
$$(1.03)^n = 1.11$$
$$\log(1.03)^n = \log(1.11)$$
$$n\log(1.03) = \log(1.11)$$
$$n = \frac{\log(1.11)}{\log(1.03)} = 3.53$$

so the firm makes a profit for the first time after 4 years.

4 **(1)** **(a)** $S = 30(1.06)^2 = \$33.71$

(b) $S = 30(1.03)^4 = \$33.77$

(c) $S = 30(1.015)^8 = \$33.79$

(d) $S = 30(1.005)^{24} = \$33.81$

(e) $S = 30(1.001\,15)^{104} = \$33.82$

(f) $S = 30(1.000\,164)^{730} = \$33.82$

(2) $S = 30e^{0.12} = \$33.82$

The results in part (1) are settling down at this value.

5

$$4000 = 1000e^{0.1r}$$
$$4 = e^{0.1r}$$
$$0.1r = \ln 4$$
$$= 1.386$$
$$r = 13.86\%$$

6 The quarterly scale factor is 1.03, so the overall scale factor for a year is

$$1.03^4 = 1.1255$$

which corresponds to a 12.55% increase.

7 After n years, the annual turnover of A will be $560(1.015)^n$ and the corresponding expression for B is $480(1.034)^n$. To find when supermarket B overtakes A we need to solve the equation

$$480(1.034)^n = 560(1.015)^n$$
$$\frac{1.034^n}{1.015^n} = \frac{560}{480}$$
$$\left(\frac{1.034}{1.015}\right)^n = \frac{7}{6}$$
$$n\log\left(\frac{1.034}{1.015}\right) = \log\left(\frac{7}{6}\right)$$
$$n = \frac{\log(7/6)}{\log(1.034/1.015)} = 8.31$$

so it will take 9 years.

Exercise 3.2 (p. 227)

1 \$6753.29; 50%

2 \$23 433.19

3 **(a)** \$619 173.64 **(b)** 13

4 15 years

5 $\$50\,000\,(0.95)^3 = \$42\,868.75$

6 **(a)** \$13 947.94 **(b)** \$14 156.59
(c) \$14 342.45 **(d)** \$14 381.03

7 \$205.44

8 36.6 years

9 17.3 years

10 We are charged interest on the interest; 26.82%.

11 7.25

12 **(a)** 6 years **(b)** 5.19%

13 \$4410; \$5143.83; 28.60%

14 21.70%

15 $P = S\left(1 + \frac{r}{100}\right)^{-n}$

16 6.17%, 7.44%, . . . , 42.58%; almost a straight line, but with a slight upward curvature.

Exercise 3.2* (p. 228)

1 13 years

2 \$158.45

3 **(a)** Midwest **(b)** BFB

4 **(a)** \$35 000 **(b)** 7 years

5 7.67%

6 **(a)** Interest is $(r/k)\%$ per period and there are kt periods in t years, so $S = P1 + \left(1 + \frac{r}{100k}\right)^{kt}$.

(b) If $m = \frac{100k}{r}$ then $\frac{r}{100k} = \frac{1}{m}$ and

$$kt = \frac{mrt}{100k} \text{ so}$$

$$S = P\left(1 + \frac{1}{m}\right)^{rtm/100} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt/100}$$

by rule 3 of indices.

(c) Now since $m = 100k/r$ we see that if the frequency increases (i.e. if $k \to \infty$) then $m \to \infty$, causing

$$\left(1 + \frac{1}{m}\right)^{m}$$

to approach e. Substituting this into the result of part **(b)** gives

$$S = Pe^{rt/100}$$

7 22 years

8 **(a)** 1.5% **(b)** 17.87%

9 **(a)** 112, 125.44, . . . , 964.63

(b) 112.55, 126.68, . . . , 1064.09

(c) 112.68, 126.97, . . . , 1089.26

(d) 112.75, 127.12, . . . , 1102.32

All four graphs have the same basic shape, and pass through (0, 100). As expected, as the frequency of compounding increases the values approach that of continuous compounding in **(d)**.

Almost a straight line, but with a slight upward curvature.

Section 3.3

Practice Problems

1 The geometric ratios of **(a)**, **(c)**, **(d)** and **(e)** are 2, −3, ½ and 1.07 respectively. The sequence in part **(b)** is an arithmetic progression, not a geometric progression, because to go from one number to the next we *add* on the fixed value of 5.

2 **(a)** The geometric ratio is 2, so the next term is $8 \times 2 = 16$.

$$1 + 2 + 4 + 8 + 16 = 31$$

For this series, $a = 1$, $r = 2$ and $n = 5$, so its value is

$$(1)\left(\frac{2^5 - 1}{2 - 1}\right) = 32 - 1 = 31$$

(b) For this series, $a = 100(1.07)$, $r = 1.07$ and $n = 20$, so its value is

$$100(1.07)\left(\frac{1.07^{20} - 1}{1.07 - 1}\right) = 4386.52$$

3 **(a)** The first \$1000 payment is invested for the full 10 years at 8% interest compounded annually, so its future value is

$$1000(1.08)^{10}$$

The second \$1000 payment is invested for 9 years, so its future value is

$$1000(1.08)^{9}$$

and so on.

The final payment of \$1000 is invested for just 1 year, so its future value is

$$1000(1.08)^{1}$$

Total savings

$$= 1000(1.08)^{10}$$
$$+ 1000(1.08)^{9}$$
$$+ \ldots + 1000(1.08)$$
$$= 1000(1.08)\left(\frac{1.08^{10} - 1}{1.08 - 1}\right)$$
$$= \$15\,645.49$$

(b) After n years,

$$\text{total savings} = 1000(1.08)\left(\frac{1.08^{n} - 1}{1.08 - 1}\right)$$
$$= 13\,500(1.08^{n} - 1)$$

so we need to solve

$$13\,500(1.08^{n} - 1) = 20\,000$$

This can be done by taking logarithms following the strategy described in Section 2.3. The corresponding value of n is 11.8, so it takes 12 years.

4 If \$$x$ denotes the monthly repayment, the amount owed at the end of the first month is

$$2000(1.01) - x$$

After 2 months the amount owed is

$$[2000(1.01) - x](1.01) - x$$
$$= 2000(1.01)^2 - x(1.01) - x$$

Each month we multiply by 1.01 to add on the interest and subtract x to deduct the repayment, so after 12 months the outstanding debt is

$$2000(1.01)^{12} - x[1.01^{11} + 1.01^{10} + \ldots + 1]$$
$$= 2253.650 - x\left(\frac{1.01^{12} - 1}{1.01 - 1}\right)$$
$$= 2253.650 - 12.683x$$

If the debt is to be cleared then

$$x = \frac{2253.650}{12.683} = \$177.69$$

5 If the annual percentage rise is 2.6%, the scale factor is 1.026, so after n years the total amount of oil extracted (in billions of units) will be

$$45.5 + 45.5(1.026) + 45.5(1.026)^2 + \ldots + 45.5(1.026)^{n-1}$$
$$= 45.5\left(\frac{1.026^n - 1}{1.026 - 1}\right) = 1750(1.026^n - 1)$$

Oil reserves are exhausted after n years where n satisfies the equation

$$1750(1.026^n - 1) = 2625$$
$$1.026^n - 1 = 1.5$$
$$n\log(1.026) = \log(2.5)$$
$$n = 35.7$$

so oil will run out after 36 years.

Exercise 3.3 (p. 238)

1 11 463.88.

2 **(a)** \$78 227.44 **(b)** \$78 941.10

3 \$983.26

4 18 years

5 200 million tonnes

6 First row: 430.33, 291.59, . . . , 96.66

Thirteenth row: 438.65, 299.36, . . . , 105.25

Exercise 3.3* (p. 239)

1 –16 777 215

2 **(a)** \$9280.71 **(b)** \$9028.14

3 140 040

4 \$424.19; **(a)** \$459.03 **(b)** \$456.44

5 \$313 238

6 $rS_n = r(a + ar + ar^2 + \ldots + ar^{n-1})$

$= ar + ar^2 + ar^3 + \ldots + ar^n$

which is very similar to the given expression for S_n except that the first term, a, is missing and we have the extra term, ar^n. Consequently, when S_n is subtracted from rS_n the rest of the terms cancel, leaving

$$rS_n - S_n = ar^n - a$$
$$(r - 1)S_n = a(r^n - 1)$$

(factorize both sides)

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

(divide through by $r - 1$)

The expression for S_n denotes the sum of the first n terms of a geometric series because the powers of r run from 0 to $n - 1$, making n terms in total. Notice that we are not allowed to divide by zero, so the last step is not valid for $r = 1$.

7 **(a)** \$480

(b) \$3024.52

(c) After n payments, debt is

$$((((8480 - A)R - A)R - A)R \ldots - A)R$$
$$= 8480R^n - AR(1 + R + R^2 + \ldots + R^{n-1})$$
$$= 8480R^n - AR\left(\frac{R^n - 1}{R - 1}\right)$$

Finally, setting this expression equal to zero gives the desired formula for A.

(d) \$637.43

8 \$3656.33; \$492 374.04, \$484 195.12, . . . , \$0

9
$$4 + 4\left(1+\frac{r}{100}\right) + 4\left(1+\frac{r}{100}\right)^2 + \cdots + 4\left(1+\frac{r}{100}\right)^{10} = 60$$
$$4\left[\frac{1-\left(1+\frac{r}{100}\right)^{10}}{1-\left(1+\frac{r}{100}\right)}\right] = 60$$
$$\frac{1-\left(1+\frac{r}{100}\right)^{10}}{-\frac{r}{100}} = 15$$
$$1-\left(1+\frac{r}{100}\right)^{10} = -0.15r$$
$$\left(1+\frac{r}{100}\right)^{10} - 0.15r - 1 = 0$$
$$r = 8.8$$

Section 3.4

Practice Problems

1 **(a)** $P = 100\,000(1.06)^{-10} = \$55\,839.48$

(b) $P = 100\,000e^{-0.6} = \$54\,881.16$

2 **(a)** $\text{NPV} = \$17\,000(1.15)^{-5} - \$8000 = \$452$

Worthwhile since this is positive.

(b) The IRR, r, is the solution of

$$8000\left(1+\frac{r}{100}\right)^5 = 17\,000$$
$$\left(1+\frac{r}{100}\right)^5 = 2.125$$

(divide by 8000)

$$1+\frac{r}{100} = 1.16$$

(take fifth roots)

$$r = 16\%$$

Worthwhile since the IRR exceeds the market rate.

3 NPV of Project A is

$$NPV_A = \$18\,000(1.07)^{-2} - \$13\,500 = \$2221.90$$

NPV of Project B is

$$NPV_B = \$13\,000(1.07)^{-2} - \$9000 = \$2354.70$$

Project B is to be preferred since

$$NPV_B > NPV_A$$

4 Rate of interest per month is $^1/_2$%, so the present value, P, of $\$S$ in t months' time is

$$P = S(1.005)^{-t}$$

The total present value is

$$2000(1.005)^{-1} + 2000(1.005)^{-2} + \ldots + 2000(1.005)^{-120}$$

because there are 120 months in 10 years. Using the formula for a geometric series gives

$$2000(1.005)^{-1}\left(\frac{1.005^{-120} - 1}{1.005^{-1} - 1}\right) = \$180\,146.91$$

5 The formula for discounting is

$$P = S(1.15)^{-t}$$

The results are given in Table S3.3.

There is very little to choose between these two projects. Both present values are considerably less than the original expenditure of $10 000. Consequently, neither project is to be recommended, since the net present values are negative. The firm would be better off just investing the $10 000 at 15% interest!

Table S3.3

	Discounted revenue ($)	
End of year	Project 1	Project 2
1	1739.13	869.57
2	1512.29	756.14
3	1972.55	1315.03
4	1715.26	3430.52
5	1491.53	1988.71
Total	8430.76	8359.97

6 The IRR satisfies the equation

$$12\,000 = 8000\left(1 + \frac{r}{100}\right)^{-1} + 2000\left(1 + \frac{r}{100}\right)^{-2} + 2000\left(1 + \frac{r}{100}\right)^{-3} + 2000\left(1 + \frac{r}{100}\right)^{-4}$$

Values of the right-hand side of this equation corresponding to $r = 5, 6, \ldots, 10$ are listed in the table below:

r	5	6	7	8	9	10
value	12 806	12 591	12 382	12 180	11 984	11 794

This shows that r is between 8 and 9. To decide which of these to go for, we evaluate $r = 8^1/_2$, which gives 12 081, which is greater than 12 000, so $r = 9\%$ to the nearest percentage. This exceeds the market rate, so the project is worthwhile.

7 If the yield is 7% then each year the income is $70 with the exception of the last year, when it is $1070 because the bond is redeemed for its original value.

The present values of this income stream are listed in Table S3.4 and are calculated using the formula

$$P = S(1.08)^{-t}$$

Table S3.4

End of year	Cash flow ($)	Present value ($)
1	70	64.81
2	70	60.01
3	1070	849.40
Total present value		974.22

Exercise 3.4 (p. 256)

1 **(a)** $5974.43 **(b)** $5965.01

2 **(a)** 7% **(b)** Yes, provided there are no risks.

3 $NPA_A = \$1595.94$; $NPV_B = \$1961.99$;
$NPV_C = \$1069.00$, so Project B is best.

4 **(a)** $379.08 **(b)** $1000

5 Project A: PV = 626.38
Project B: PV = 1248.28
Choose B

6 20.3%

Exercise 3.4* (p. 256)

1 $257.85

2 $5000

3 $31 250

4 $61 672.67

5 $38 887.69

6 27%

7 $349.15

8 **(a)** $400 000 **(b)** $92 550.98 **(c)** $307 449.02

9 **(a)** \$333 million

(b) $5\left(\frac{1-(1.06)^{1-n}}{1.06^2-1.06}\right)$

(c) Same expression as **(b)** but with 5 replaced by 50.

(d) 12

10 \$49 280

11 $R(1+r/100)^{-1}+R(1+r/100)^{-2}+\ldots+R(1+r/100)^{-n}$

$$=\frac{R}{1+r/100}\left(\frac{1-(1+r/100)^{-n}}{1-(1+r/100)^{-1}}\right)$$

$$=R\left(\frac{1-(1+r/100)^{-n}}{1+r/100-1}\right)$$

$$=100R\left(\frac{1-(1+r/100)^{-n}}{r}\right)$$

(a) \$1488.94

(b) $\frac{100R}{r}$

12 12% to the nearest whole number and 11.6% to 1 decimal place.

13 \$22 177 and \$16 659, so choose A.

\$21 702 and \$16 002, so very little difference in NPVs and no difference in choice.

Chapter 4

Section 4.1

Practice Problems

1 **(a)** $\frac{11-3}{3-(-1)}=\frac{8}{4}=2$

(b) $\frac{-2-3}{4-(-1)}+\frac{-5}{5}=-1$

(c) $\frac{3-3}{49-(-1)}=\frac{0}{50}=0$

2 Using a calculator, the values of the cube function, correct to 2 decimal places, are

x	−1.50	−1.25	−1.00	−0.75
$f(x)$	−3.38	−1.95	−1.00	−0.42

x	−0.50	−0.25	0.00	0.25	0.50
$f(x)$	−0.13	−0.02	0.00	0.02	0.13

x	0.75	1.00	1.25	1.50
$f(x)$	0.42	1.00	1.95	3.38

The graph of the cube function is sketched in Figure S4.1.

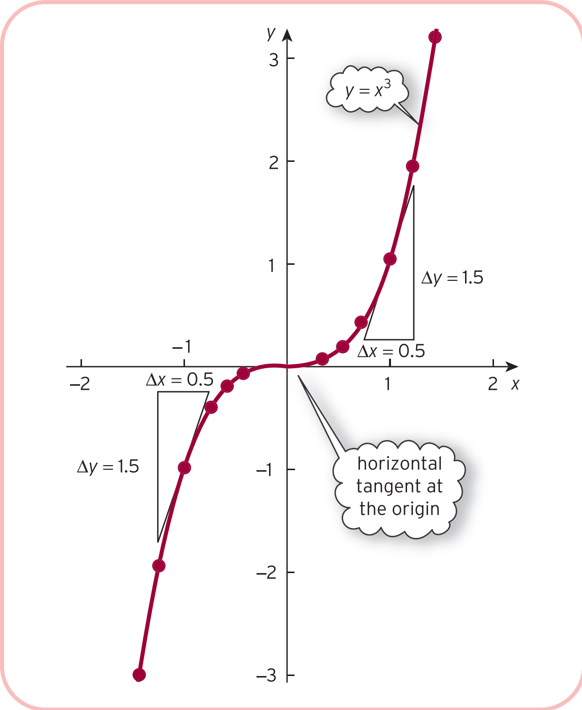


Figure S4.1

$$f'(-1)=\frac{1.5}{0.5}=3.0$$

$f'(0)=0$ (because the tangent is horizontal at $x=0$)

$$f'(1)=\frac{1.5}{0.5}=3.0$$

[Note: $f'(-1)=f'(1)$ because of the symmetry of the graph.]

3 If $n=3$ then the general formula gives

$f'(x)=3x^{3-1}=3x^2$

Hence

$f'(-1)=3(-1)^2=3$

$f'(0)=3(0)^2=0$

$f'(1)=3(1)^2=3$

4 **(a)** $5x^4$ **(b)** $6x^5$ **(c)** $100x^{99}$

(d) $-x^{-2}$ (that is, $-1/x^2$) **(e)** $-2x^{-3}$ (that is, $-2/x^3$)

Exercise 4.1 (p. 273)

1 **(a)** 2 **(b)** −1 **(c)** 0

2 $-^2/_3$; downhill

3 The graph of $f(x)=5$ is sketched in Figure S4.2. The graph is horizontal, so has zero slope at all values of x.

4 $7x^6$; 448

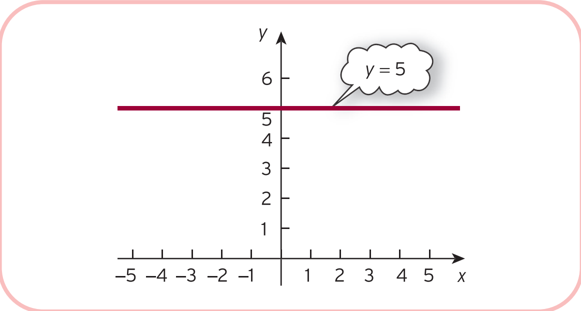


Figure S4.2

5 (a) $8x^7$ (b) $50x^{49}$ (c) $19x^{18}$ (d) $999x^{998}$

6 (a) $\frac{-3}{x^4}$ (b) $\frac{1}{2\sqrt{x}}$ (c) $-\frac{1}{2x\sqrt{x}}$ (d) $\frac{3\sqrt{x}}{2}$

7 3, 1.25, 0, –0.75, –1, –0.75, 0, 1.25

(a) –3 (b) 0 (c) 1

Exercise 4.1* (p. 274)

1 When $x = 0$, $y = a(0) + b = b$ ✓

When $x = 1$, $y = a(1) + b = a + b$ ✓

$$\text{Slope} = \frac{(a+b)-b}{1-0} = a$$

2 (a) $15x^{14}$ (b) $\frac{9x^3\sqrt{x}}{2}$ (c) $\frac{\sqrt[3]{x}}{3x}$

(d) $\frac{-1}{4\sqrt[4]{x}x}$ (e) $\frac{-13\sqrt{x}}{2x^8}$

3 (a) (i) 2, 2.024 8 . . . (ii) 0.248 45 . . . (iii) 0.25

(b) (i) 8, 8.301 8 . . . (ii) 3.018 67 . . . (iii) 3

(c) (i) 0.5, 0.498 6 . . . (ii) –0.061 35 . . . (iii) –0.0625

In all three cases the gradient of the chord gives a good approximation to that of the tangent.

4 (a) (8, 4) (b) (±3, ±243) (c) $-\left(-\frac{1}{2}, \frac{1}{4}\right)$ (d) $\left(4, \frac{1}{8}\right)$

Section 4.2

Practice Problems

1 (a) $4(3x^2) = 12x^2$

(b) $-2/x^3$ because $1/x = x^{-1}$, which differentiates to $-x^{-2}$.

2 (a) $5x^4 + 1$ (b) $2x + 0 = 2x$

3 (a) $2x - 3x^2$ (b) $0 - (-3x^{-4}) = \frac{3}{x^4}$

4 (a) $9(5x^4) + 2(2x) = 45x^4 + 4x$

(b) $5(8x^7) - 3(-1)x^{-2} = 40x^7 + 3/x^2$

(c) $2x + 6(1) + 0 = 2x + 6$

(d) $2(4x^3) + 12(3x^2) - 4(2x) + 7(1) - 0$
$= 8x^3 + 36x^2 - 8x + 7$

5 $f'(x) = 4(3x^2) - 5(2x) = 12x^2 - 10x$

$f''(x) = 12(2x) - 10(1) = 24x - 10$

$f''(6) = 24(6) - 10 = 134$

Exercise 4.2 (p. 282)

1 (a) $10x$ (b) $-3/x^2$ (c) 2 (d) $2x + 1$

(e) $2x - 3$ (f) $3 + 7/x^2$ (g) $6x^2 - 12x + 49$

(h) a (i) $2ax + b$ (j) $2/\sqrt{x} + 3/x^2 - 14/x^3$

2 (a) 27 (b) 4 (c) 2 (d) –36 (e) 3/8

3 $4x^3 + 6x$

(a) $9x^2 - 8x$ (b) $12x^3 - 6x^2 + 12x - 7$ (c) $2x - 5$

(d) $1 + \frac{3}{x^2}$ (e) $-\frac{2}{x^3} + \frac{4}{x^2}$ (f) $\frac{3}{x^2} - \frac{10}{x^3}$

4 (a) 14 (b) $6/x^4$ (c) 0

5 4

6 0; horizontal tangent, i.e. vertex of parabola must be at $x = 3$.

7 $\frac{1}{\sqrt{x}}$ (a) $\frac{5}{2\sqrt{x}}$ (b) $x^{-2/3}$ (c) $\frac{3}{2}x^{-1/4}$ (d) $-\frac{5}{2}x^{-3/2}$

8 (a) $2P + 1$ (b) $50 - 6Q$ (c) $-30/Q^2$

(d) 3 (e) $5/\sqrt{L}$ (f) $-6Q^2 + 30Q - 24$

Exercise 4.2* (p. 284)

1 $\frac{3}{2}$

2 (a) $4P + 1$ (b) $40 - \frac{9}{2}\sqrt{Q}$ (c) $-\frac{20}{Q^2} + 7$

(d) $4Y + 3$ (e) $200 + \frac{1}{L\sqrt[4]{L}}$ (f) $-3Q^2 + 40Q - 7$

3 –40; graph bends downwards.

4 (a) Uphill; $f'(-1) = 1 > 0$

(b) Bends downwards; $f''(-1) = -72 < 0$

5 $f''(x) = 6ax + 2b > 0$ gives $x > -b/3a$

$f''(x) = 6ax + 2b > 0$ gives $x < -b/3a$

6 $y = x - 3$

Section 4.3

Practice Problems

1 $TR = PQ = (60 - Q)Q = 60Q - Q^2$

(1) $MR = 60 - 2Q$

When $Q = 50$

$MR = 60 - 2(50) = -40$

(2) (a) $TR = 60(50) - (50)^2 = 500$

(b) $TR = 60(51) - (51)^2 = 459$

so TR changes by –41, which is approximately the same as the exact value obtained in part (1).

2 MR = 1000 − 8Q, so when Q = 30

MR = 1000 − 8(30) = 760

(a) $\Delta(\text{TR}) \simeq \text{MR} \times \Delta Q = 760 \times 3 = 2280$, so total revenue rises by about 2280.

(b) $\Delta(\text{TR}) \simeq \text{MR} \times \Delta Q = 760 \times (-2) = -1520$

so total revenue falls by about 1520.

3 $\text{TC} = (\text{AC})Q = \left(\frac{100}{Q} + 2\right)Q = 100 + 2Q$

This function differentiates to give MC = 2, so a 1 unit increase in Q always leads to a 2 unit increase in TC irrespective of the level of output.

4 If K = 100 then

$Q = 5L^{1/2}(100)^{1/2} = 50L^{1/2}$

because $\sqrt{100} = 10$. Differentiating gives

$\text{MP}_L = 50(\tfrac{1}{2}L^{-1/2}) = \frac{25}{\sqrt{L}}$

(a) $\frac{25}{\sqrt{1}} = 25$ (b) $\frac{25}{\sqrt{9}} = \frac{25}{3} = 8.3$

(c) $\frac{25}{\sqrt{10\,000}} = \frac{25}{100} = 0.25$

The fact that these values decrease as L increases suggests that the law of diminishing marginal productivity holds for this function. This can be confirmed by differentiating a second time to get

$$\frac{d^2Q}{dL^2} = 25(-1/2L^{-3/2}) = \frac{-25}{2L^{3/2}}$$

which is negative for all values of L.

5 The savings function is given, so we begin by finding MPS. Differentiating S with respect to Y gives

MPS = 0.04Y − 1

so when Y = 40,

MPS = 0.04(40) − 1 = 0.6

To find MPC we use the formula

MPC + MPS = 1

that is,

MPC = 1 − MPS = 1 − 0.6 = 0.4

This indicates that, at the current level of income, a 1 unit increase in national income causes a rise of about 0.6 units in savings and 0.4 units in consumption.

Exercise 4.3 (p. 297)

1 TR = 100Q − 4Q^2, MR = 100 − 8Q; 1.2

2 TR = 80Q − 3Q^2, so MR = 80 − 6Q = 80 − 6(80 − P)/3 = 2P − 80

3 TR = 100Q − Q^2; MR = 100 − 2Q. Graphs of TR and MR are sketched in Figures S4.3 and S4.4 respectively. MR = 0 when Q = 50. This is the value of Q at which TR is a maximum.

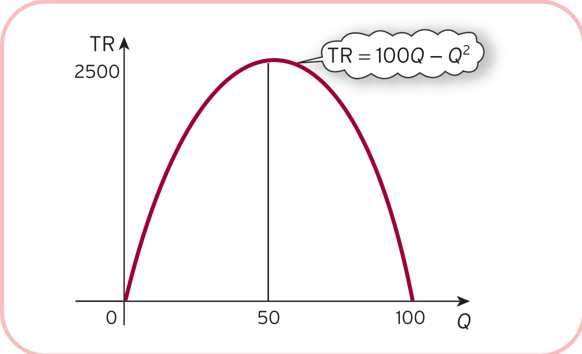


Figure S4.3

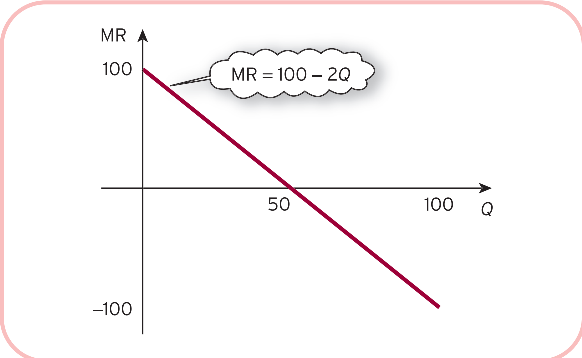


Figure S4.3

4 TC = 15 + 2Q^2 + 9Q; 15; 4Q + 9

5 (a) 49.98 (b) 49.8 (c) 48 (d) 30

Yes, because $d^2Q/dL^2 = -0.02 < 0$.

6 MPC = 1/6 and MPS = 5/6. If national income rises by 1 unit, the approximate increase in consumption and savings is 1/6 and 5/6, respectively.

7 13

Exercise 4.3* (p. 298)

1 (a) TR = 100Q − 4$Q^{3/2}$ − 3Q^2

(b) MR = 100 − 6$Q^{1/2}$ − 6Q; MR = 28

(c) 7 compared to 6.78

2 (a) MPC = 0.96, MPS = 0.04

(b) $S = 0.2Y - 100 - 0.01Y^2$

3 (a) TC = 100 + 2Q + Q^2/10

MC = 2 + Q/5

(b) MC = 8; $\Delta(\text{TC}) \simeq 16$

(c) 100

4 $\frac{d^2Q}{dL^2} = 12 - 1.2L < 0$ for all $L > 10$.

5 **(a)** $MP_L = \frac{5}{2}L^{-1/2} - 0.1$

(b) $L = 625$; output is maximized when $L = 625$.

(c) $\frac{d^2Q}{dL^2} = -\frac{5}{4}L^{-3/2} < 0$

6 36

Section 4.4

Practice Problems

1 **(a)** The outer power function differentiates to get $5(3x - 4)^4$ and the derivative of the inner function, $3x - 4$, is 3, so

$$\frac{dy}{dx} = 5(3x - 4)^4(3) = 15(3x - 4)^4$$

(b) The outer power function differentiates to get $3(x^2 + 3x + 5)^2$ and the derivative of the inner function, $x^2 + 3x + 5$, is $2x + 3$, so

$$\frac{dy}{dx} = 3(x^2 + 3x + 5)^2(2x + 3)$$

(c) Note that $y = (2x - 3)^{-1}$. The outer power function differentiates to get $-(2x - 3)^{-2}$ and the derivative of the inner function, $2x - 3$, is 2, so

$$\frac{dy}{dx} = -(2x - 3)^{-2}(2) = \frac{-2}{(2x - 3)^2}$$

(d) Note that $y = (4x - 3)^{1/2}$. The outer power function differentiates to get $\frac{1}{2}(4x - 3)^{-1/2}$ and the derivative of the inner function, $4x - 3$, is 4, so

$$\frac{dy}{dx} = \tfrac{1}{2}(4x - 3)^{-1/2}(4) = \frac{2}{\sqrt{(4x - 3)}}$$

2 **(a)** $u = x \qquad v = (3x - 1)^6$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 6(3x - 1)^5(3)$$

So

$$\frac{dy}{dx} = 18x(3x - 1)^5 + (3x - 1)^6$$
$$= (3x - 1)^5[18x + (3x - 1)]$$
$$= (3x - 1)^5(21x - 1)$$

(b) $u = x^3 \quad v = (2x + 3)^{1/2}$

$$\frac{du}{dx} = 3x^2 \quad \frac{dv}{dx} = \tfrac{1}{2}(2x + 3)^{-1/2}(2)$$
$$= \frac{1}{\sqrt{(2x + 3)}}$$

So

$$\frac{dy}{dx} = \frac{x^3}{\sqrt{(2x + 3)}} + 3x^2\sqrt{(2x + 3)}$$

(c) $u = x \qquad v = (x - 2)^{-1}$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = -(x - 2)^{-2}$$

So

$$\frac{dy}{dx} = \frac{-x}{(x - 2)^2} + \frac{1}{x - 2}$$
$$= \frac{-x + (x - 2)}{(x - 2)^2}$$
$$= \frac{-2}{(x - 2)^2}$$

3 **(a)** $u = x \qquad v = x - 2$

$$\frac{dy}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

So

$$\frac{dy}{dx} = \frac{(x - 2) - x}{(x - 2)^2}$$
$$= \frac{-2}{(x - 2)^2}$$

(b) $u = x - 1 \qquad v = x + 1$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1$$

So

$$\frac{dy}{dx} = \frac{(x + 1) - (x - 1)}{(x + 1)^2}$$
$$= \frac{2}{(x + 1)^2}$$

Exercise 4.4 (p. 307)

1 **(a)** $15(5x + 1)^2$ **(b)** $16(2x - 7)^7$ **(c)** $5(x + 9)^4$

(d) $24x(4x^2 - 7)^2$ **(e)** $8(x + 2)(x^2 + 4x - 3)$

(f) $\frac{1}{\sqrt{2x + 1}}$ **(g)** $\frac{-3}{(3x + 1)^2}$ **(h)** $\frac{-8}{(4x - 3)^3}$

(i) $-(2x + 3)^{-3/2} = \frac{-1}{(2x + 3)\sqrt{(2x + 3)}}$

2 **(a)** $(3x + 4)^2 + 6x(3x + 4) = (9x + 4)(3x + 4)$

(b) $2x(x - 2)^3 + 3x^2(x - 2)^2 = x(5x - 4)(x - 2)^2$

(c) $\sqrt{x + 2} + \frac{x}{2\sqrt{x + 2}} = \frac{3x + 4}{2\sqrt{x + 2}}$

(d) $(x + 6)^3 + 3(x - 1)(x + 6)^2 = (4x + 3)(x + 6)^2$

(e) $2(x + 5)^3 + 3(2x + 1)(x + 5)^2 = (8x + 13)(x + 5)^2$

(f) $3x^2(2x - 5)^4 + 8x^3(2x - 5)^3 = x^2(14x - 15)(2x - 5)^3$

3 (a) $\frac{-5}{(x-5)^2}$ (b) $\frac{7}{(x+7)^2}$

(c) $\frac{-5}{(x-2)^2}$ (d) $\frac{-25}{(3x+1)^2}$

(e) $\frac{6}{(5x+6)^2}$ (f) $\frac{-19}{(3x-7)^2}$

4 $10(5x+7) = 50x + 70$

5 $5x^4(x+2)^2 + 2x^5(x+2) = 7x^6 + 24x^5 + 20x^4$

6 (a) $(100 - 4Q)(100 - Q)^2$

(b) $\frac{4000}{(Q+4)^2}$

7 MPC = 1.8, MPS = −0.8. If national income rises by 1 unit, consumption rises by 1.8 units, whereas savings actually fall by 0.8 units.

Exercise 4.4* (p. 308)

1 (a) $20(2x+1)^9$

(b) $3(x^2 + 3x - 5)^2(2x + 3)$

(c) $-7/(7x-3)^2$

(d) $-2x/(x^2+1)^2$

(e) $4/\sqrt{(8x-1)}$

(f) $-2(6x-5)^{-4/3} = \frac{-2}{(6x-5)\sqrt[3]{6x-5}}$

2 (a) $5x(x+2)(x+5)^2$

(b) $x^4(4x+5)(28x+25)$

(c) $\frac{x^3(9x+8)}{2\sqrt{(x+1)}}$

3 (a) $\frac{x^3+8x}{(x+4)^2}$ (b) $\frac{3}{(x+1)^2}$ (c) $\frac{x^2(5x-6)}{2(x-1)^{3/2}}$

4 (a) $(x-3)^3(5x-3)$ (b) $\frac{3x-3}{\sqrt{2x-3}}$ (c) $\frac{3x^2(x+5)}{(3x+5)^3}$

(d) $\frac{1-x^2}{(x^2+1)^2}$ (e) $\frac{ad-bc}{(cx+d)^2}$

(f) $[ac(m+n)x + mad + ncb](ax+b)^{m-1}(cx+d)^{n-1}$

(g) $(6x^2 + 17x + 6)(x+2)(x+3)^2$

5 $\frac{-4}{(2x+1)^3}$

6 (a) $\frac{100-3Q}{\sqrt{(100-2Q)}}$

(b) $\frac{2000+500Q}{(2+Q)^{3/2}}$

7 1.098; −0.098; if income rises by 1 unit, consumption goes up by more than this, with the excess taken out of savings.

Section 4.5

Practice Problems

1 We are given that $P_1 = 210$ and $P_2 = 200$. Substituting $P = 210$ into the demand equation gives

$$1000 - 2Q_1 = 210$$
$$-2Q_1 = -790$$
$$Q_1 = 395$$

Similarly, putting $P = 200$ gives $Q_2 = 400$. Hence

$$\Delta P = 200 - 210 = -10$$
$$\Delta Q = 400 - 395 = 5$$

Averaging the P values gives

$$P = {}^1/_2(210 + 200) = 205$$

Averaging the Q values gives

$$Q = {}^1/_2(395 + 400) = 397.5$$

Hence, arc elasticity is

$$-\left(\frac{205}{397.5}\right) \times \left(\frac{5}{-10}\right) = 0.26$$

2 The quickest way of solving this problem is to find a general expression for E in terms of P and then just to replace P by 10, 50 and 90 in turn. The equation

$$P = 100 - Q$$

rearranges as

$$Q = 100 - P$$

so

$$\frac{dQ}{dP} = -1$$

Hence

$$E = -\frac{P}{Q} \times \frac{dQ}{dP} = \frac{-P}{100-P} \times (-1)$$
$$= \frac{P}{100-P}$$

(a) If $P = 10$ then $E = 1/9 < 1$ so inelastic.

(b) If $P = 50$ then $E = 1$ so unit elastic.

(c) If $P = 90$ then $E = 9$ so elastic.

At the end of Section 4.5 it is shown quite generally that the price elasticity of demand for a linear function

$$P = aQ + b$$

is given by

$$E = \frac{P}{b-P}$$

The above is a special case of this with $b = 100$.

3 Substituting $Q = 4$ into the demand equation gives

$$P = -(4)^2 - 10(4) + 150 = 94$$

Differentiating the demand equation with respect to Q gives

$$\frac{dP}{dQ} = -2Q - 10$$

so

$$\frac{dQ}{dP} = \frac{1}{-2Q - 10}$$

When $Q = 4$

$$\frac{dQ}{dP} = -\frac{1}{18}$$

The price elasticity of demand is then

$$-\left(\frac{94}{4}\right) \times \left(-\frac{1}{18}\right) = \frac{47}{36}$$

From the definition

$$E = -\frac{\text{percentage change in demand}}{\text{percentage change in price}}$$

we have

$$\frac{47}{36} = -\frac{10}{\text{percentage change in price}}$$

Hence the percentage change in price is $-10 \times 36/47 = -7.7\%$: that is, the firm must reduce prices by 7.7% to achieve a 10% increase in demand.

4 **(a)** Putting $P = 9$ and 11 directly into the supply equation gives $Q = 203.1$ and 217.1 respectively, so

$$\Delta P = 11 - 9 = 2$$

$$\Delta Q = 217.1 - 203.1 = 14$$

Averaging the P values gives

$$P = {}^1/_2(9 + 11) = 10$$

Averaging the Q values gives

$$Q = {}^1/_2(203.1 + 217.1) = 210.1$$

Arc elasticity is

$$\frac{10}{210.1} \times \frac{14}{2} = 0.333\ 175$$

(b) Putting $P = 10$ directly into the supply equation, we get $Q = 210$. Differentiating the supply equation immediately gives

$$\frac{dQ}{dP} = 5 + 0.2P$$

so when $P = 10$, $dQ/dP = 7$. Hence

$$E = \frac{10}{210} \times 7 = \frac{1}{3}$$

Note that, as expected, the results in parts (a) and (b) are similar. They are not identical, because in part (a) the elasticity is 'averaged' over the arc from $P = 9$ to $P = 11$, whereas in part (b) the elasticity is evaluated exactly at the midpoint, $P = 10$.

Exercise 4.5 (p. 322)

1 43/162 = 0.27

2 22/81 = 0.27; agree to 2 decimal places.

3 **(a)** 1/4 **(b)** 1/4 **(c)** 9/8

4 **(a)** $0.2P$

(b) $0.1P^2 = Q - 4$

(subtract 4 from both sides)

$$P^2 = 10(Q - 4) = 10Q - 40$$

(multiply both sides by 10)

$$P = \sqrt{(10Q - 40)}$$

(square root both sides)

$$\frac{dP}{dQ} = \frac{5}{\sqrt{(10Q - 40)}}$$

(c) $\frac{1}{dP/dQ} = \frac{\sqrt{(10Q - 40)}}{5} = \frac{P}{5} = 0.2P = \frac{dQ}{dP}$

(d) $E = 10/7$

5 1.46; **(a)** elastic **(b)** 7.3%

Exercise 4.5* (p. 322)

1 1.54

2 0.25%

3 $4P/(60 - 4P)$; 7.5

4 If $P = AQ^{-n}$ then

$$\frac{dP}{dQ} = -nAQ^{-(n+1)}$$

so

$$\frac{dQ}{dP} = \frac{1}{-nAQ^{-(n+1)}}$$

Hence

$$E = -\frac{P}{Q} \times \frac{dQ}{dP} = -\left(\frac{AQ^{-n}}{Q}\right) \times \left(\frac{1}{-nAQ^{-(n+1)}}\right)$$

$$= -Q^{-n} \times \left(\frac{1}{-nQ^{-n}}\right) = \frac{1}{n}$$

which is a constant.

5 $E = \frac{P}{Q} \times \frac{dQ}{dP} = \frac{P}{Q} \times a = \frac{Pa}{Q} = \frac{aP}{aP + b}$

(a) if $b = 0$ then $E = \frac{Pa}{Pa} = 1$

(b) if $b > 0$ then $aP + b > aP$ so $E = \frac{aP}{aP + b} < 1$

Assuming that the line is sketched with quantity on the horizontal axis and price on the vertical axis, supply is unit elastic when the graph passes through the origin, and inelastic when the vertical intercept is positive.

6 (1) 0.528

(2) $\frac{0.2P^2}{40 + 0.1P^2}$; (a) 4.2% (b) 20

7 (a) $Q = \frac{P - b}{a} \Rightarrow \frac{dQ}{dP} = \frac{1}{a}$

$$E = P \times \frac{a}{P - b} \times \frac{1}{a} = \frac{P}{P - b}$$

(b) $a = \frac{2}{5}$, $b = 9$

Section 4.6

Practice Problems

1 (a) Step 1

$$\frac{dy}{dx} = 6x + 12 = 0$$

has solution $x = -2$.

Step 2

$$\frac{d^2y}{dx^2} = 6 > 0$$

so minimum.

Finally, note that when $x = -2$, $y = -47$, so the minimum point has coordinates $(-2, -47)$.
A graph is sketched in Figure S4.5.

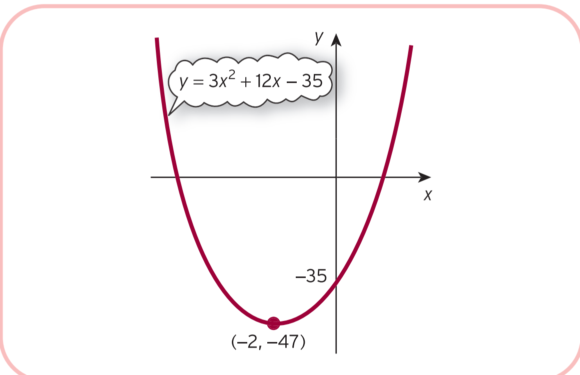


Figure S4.5

(b) Step 1

$$\frac{dy}{dx} = -6x^2 + 30x - 36 = 0$$

has solutions $x = 2$ and $x = 3$.

Step 2

$$\frac{d^2y}{dx^2} = -12x + 30x$$

which takes the values 6 and −6 at $x = 2$ and $x = 3$ respectively. Hence minimum at $x = 2$ and maximum at $x = 3$.

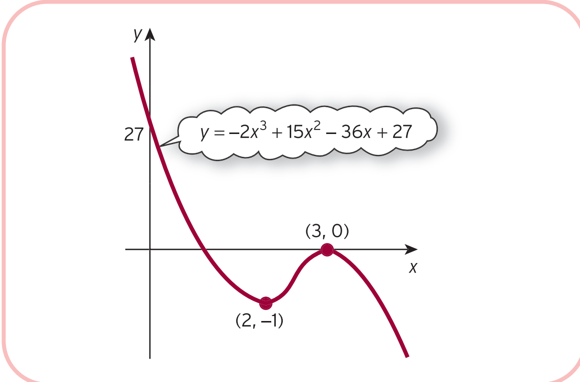


Figure S4.6

A graph is sketched in Figure S4.6 based on the following table of function values:

x	−10	0	2	3	10
$f(x)$	3887	27	−1	0	−833

2 $AP_L = \frac{Q}{L} = \frac{300L^2 - L^4}{L} = 300L - L^3$

Step 1

$$\frac{d(AP_L)}{dL} = 300 - 3L^2 = 0$$

has solution $L = \pm 10$. We can ignore −10 because it does not make sense to employ a negative number of workers.

Step 2

$$\frac{d^2(AP_L)}{dL^2} = -6L$$

which takes the value $-60 < 0$ at $L = 10$. Hence $L = 10$ is a maximum.

Now

$$MP_L = \frac{dQ}{dL} = 600L - 4L^3$$

so at $L = 10$

$$MP_L = 600(10) - 4(10)^3 = 2000$$

$$AP_L = 300(10) - (10)^3 = 2000$$

that is, $MP_L = AP_L$.

3 **(a)** $TR = PQ = (20 - 2Q)Q = 20Q - 2Q^2$

Step 1

$$\frac{d(TR)}{dQ} = 20 - 4Q = 0$$

has solution $Q = 5$.

Step 2

$$\frac{d^2(TR)}{dQ^2} = -2 < 0$$

so maximum.

(b) $\pi = TR - TC$

$$= (20Q - 2Q^2) - (Q^3 - 8Q^2 + 20Q + 2)$$
$$= -Q^3 + 6Q^2 - 2$$

Step 1

$$\frac{d\pi}{dQ} = -3Q^2 + 12Q = 0$$

has solutions $Q = 0$ and $Q = 4$.

Step 2

$$\frac{d^2\pi}{dQ^2} = -6Q + 12$$

which takes the values 12 and −12 when $Q = 0$ and $Q = 4$, respectively. Hence minimum at $Q = 0$ and maximum at $Q = 4$.

Finally, evaluating π at $Q = 4$ gives the maximum profit, $\pi = 30$. Now

$$MR = \frac{d(TR)}{dQ} = 20 - 4Q$$

so at $Q = 4$, MR = 4.

$$MC = \frac{d(TC)}{dQ} = 3Q^2 - 16Q + 20$$

so at $Q = 4$, MC = 4.

4 $AC = Q + 3 + \frac{36}{Q}$

Step 1

$$\frac{d(AC)}{dQ} = 1 - \frac{36}{Q^2} = 0$$

has solution $Q = \pm 6$. A negative value of Q does not make sense, so we just take $Q = 6$.

Step 2

$$\frac{d^2(AC)}{dQ^2} = \frac{72}{Q^3}$$

is positive when $Q = 6$, so it is a minimum.

Now when $Q = 6$, AC = 15. Also

$$MC = \frac{d(TC)}{dQ} = 2Q + 3$$

which takes the value 15 at $Q = 6$. We observe that the values of AC and MC are the same: that is, at the point of minimum average cost

average cost	=	marginal cost

There is nothing special about this example and in the next section we show that this result is true for any average cost function.

5 After tax the supply equation becomes

$$P = \tfrac{1}{2}Q_S + 25 + t$$

In equilibrium, $Q_S = Q_D = Q$, so

$$P = \tfrac{1}{2}Q + 25 + t$$
$$P = -2Q + 50$$

Hence

$$\tfrac{1}{2}Q + 25 + t = -2Q + 50$$

which rearranges to give

$$Q = 10 - \tfrac{2}{5}t$$

Hence the tax revenue, T, is

$$T = tQ = 10t - \tfrac{2}{5}t^2$$

Step 1

$$\frac{dT}{dt} = 10 - \tfrac{4}{5}t = 0$$

has solution $t = 12.5$.

Step 2

$$\frac{d^2T}{dt^2} = \frac{-4}{5} < 0$$

so maximum. Government should therefore impose a tax of \$12.50 per good.

Exercise 4.6 (p. 342)

1 **(a)** Maximum at (1/2, 5/4); graph is sketched in Figure S4.7.

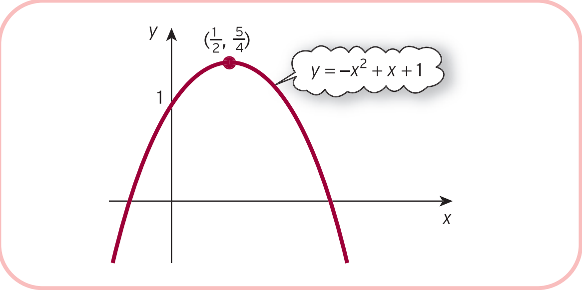


Figure S4.7

(b) Minimum at (2, 0); graph is sketched in Figure S4.8.

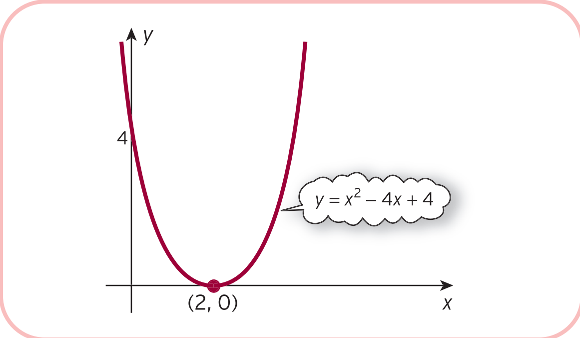


Figure S4.8

(c) Minimum at (10, 5); graph is sketched in Figure S4.9.

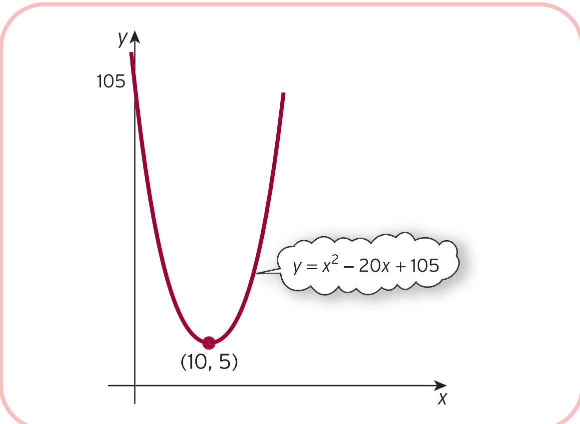


Figure S4.9

(d) Maximum at (1, 2), minimum at (–1, –2); graph is sketched in Figure S4.10.

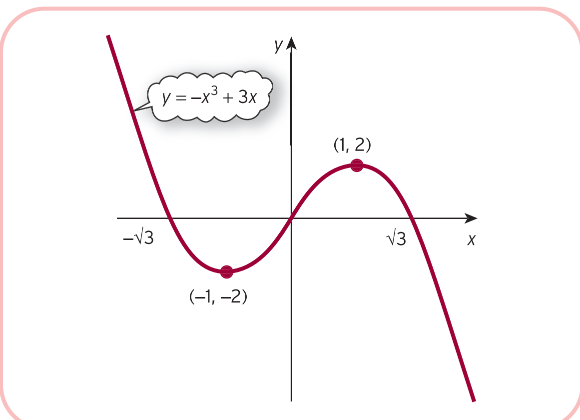


Figure S4.10

2 10

3 30; $MP_L = 450 = AP_L$

4 **(a)** $TC = 13 + (Q + 2)Q$

$$= 13 + Q^2 + 2Q$$

$$AC = \frac{TC}{Q} = \frac{13}{Q} + Q + 2$$

Q	1	2	3	4	5	6
AC	16	10.5	9.3	9.3	9.6	10.2

The graph of AC is sketched in Figure S4.11.

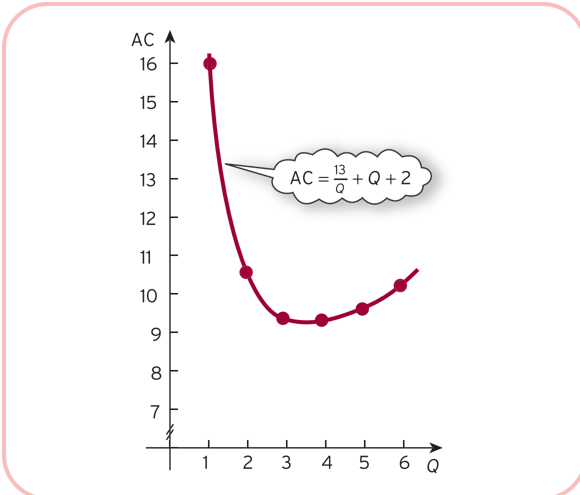


Figure S4.11

(b) From Figure S4.11 minimum average cost is 9.2.

(c) Minimum at $Q = \sqrt{13}$, which gives AC = 9.21.

5 **(a)** $TR = 4Q - \frac{Q^2}{4}$

$$\pi = \frac{-Q^3}{20} + \frac{Q^2}{20} + 2Q - 4$$

$$MR = 4 - \frac{Q}{2}$$

$$MC = 2 - \frac{3Q}{5} + \frac{3Q^2}{20}$$

(b) 4

(c) MR = 2 = MC

6 \$3

7 **(a)** $TC = 2Q^2 - 36Q + 200$

$$AC = \frac{TC}{Q} = 2Q - 36 + \frac{200}{Q}$$

(b) $Q = 10 \quad \Rightarrow \quad AC = 4$

At $Q = 10$, $\frac{d^2(AC)}{dQ^2} = 0.4 > 0 \Rightarrow$ minimum

(c) $Q = 10 \Rightarrow MC = 4 = AC$

8 (a) Minimum at (1, 7) and (4, −34), maximum at (2, −2).

(b) Minimum at (0, −10), inflection at (3, 17).

(c) Minimum at (−1, −½), maximum at (1, ½).

9 $$\pi = -30Q + \frac{13}{2}Q^2 - \frac{1}{3}Q^3$$

$$MC = 80 - 15Q + Q^2$$

$$MR = 50 - 2Q$$

Exercise 4.6* (p. 344)

1 531.5

2 Graphs of the three functions are sketched in Figure S4.12, which shows that the stationary points in (a), (b) and (c) are a point of inflection, minimum and maximum, respectively.

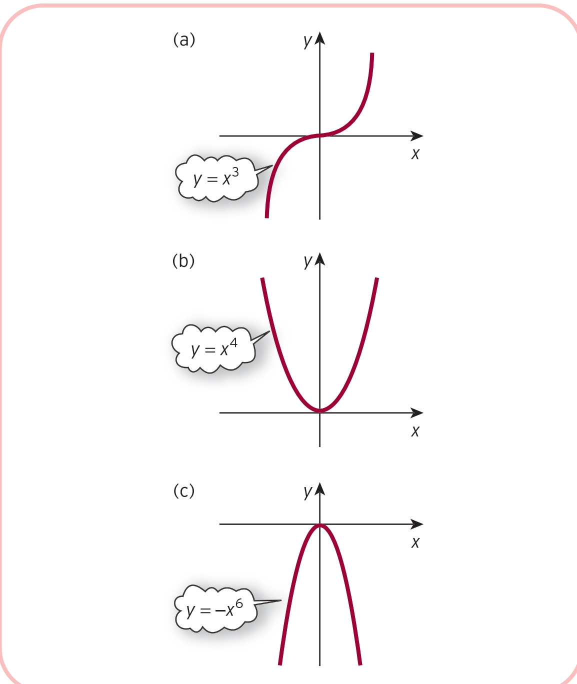


Figure S4.12

3 $TC = 2Q^2 + 15$, $AC = 2Q + \frac{15}{Q}$, $MC = 4Q$; $\sqrt{7.5}$;

$AC = 11 = MC$

4 (a) 37 037 after 333 days

(b) 167

5 167

6 (a) 15

(b) $Q = 15 - \frac{t}{10}$ gives $P = 140 + \frac{2t}{5}$

P rises from 140 to $140 + \frac{2t}{5}$, so the increase is $\frac{2}{5}t$.

7 $a = -7$, $b = 16$, $c = -7$

8 (a) Maple chooses a poor range on the y axis so that you can barely distinguish the graph from the axes. This is because the graph is undefined at $x = 0$, resulting in very large (positive and negative) values of y when x is close to 0.

(c) Minimum at (1, 1), maximum at (5, 1/9).

9 29.54

Section 4.7

Practice Problems

1 (a) $TR = (25 - 0.5Q)Q = 25Q - 0.5Q^2$

$TC = 7 + (Q + 1)Q = Q^2 + Q + 7$

$MR = 25 - Q$

$MC = 2Q + 1$

(b) From Figure S4.13 the point of intersection of the MR and MC curves occurs at $Q = 8$. The MC curve cuts the MR curve from below, so this must be a maximum point.

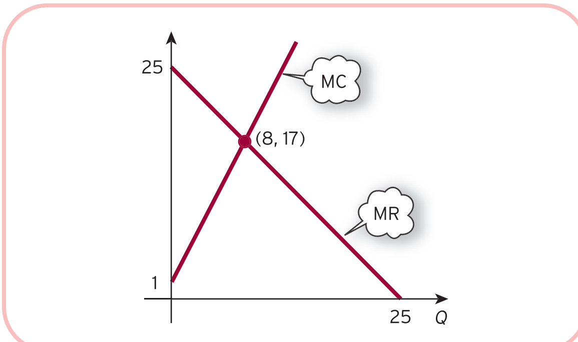


Figure S4.13

2 $MC = 100$.

(a) *Domestic market* $P_1 = 300 - Q_1$

$TR_1 = 300Q_1 - Q_1^2$

so

$MR_1 = 200 - 2Q_1$

To maximize profit, $MR_1 = MC$: that is,

$300 - 2Q_1 = 100$

which has solution $Q_1 = 100$.

Corresponding price is

$P_1 = 300 - 100 = \$200$

Foreign market $P_2 = 200 - \frac{1}{2}Q_2$

$TR_2 = 200Q_2 - \frac{1}{2}Q_2^2$

so

$MR_2 = 200 - Q_2$

To maximize profit, $MR_2 = MC$: that is,

$200 - Q_2 = 100$

which has solution $Q_2 = 100$.

Corresponding price is

$P_2 = 200 - \frac{1}{2}(100) = \$150$

(b) Without discrimination, $P_1 = P_2 = P$, say, so individual demand equations become

$Q_1 = 300 - P$

$Q_2 = 400 - 2P$

Adding shows that the demand equation for combined market is

$Q = 700 - 3P$

where $Q = Q_1 + Q_2$.

$$TR = \frac{700}{3}Q - \frac{Q^2}{3}$$

so

$$MR = \frac{700}{3} - \frac{2Q}{3}$$

To maximize profit, MR = MC: that is,

$$\frac{700}{3} - \frac{2Q}{3} = 100$$

which has solution $Q = 200$.

Corresponding price is

$P = 700/3 - 200/3 = \$500/3$

Total cost of producing 200 goods is

$5000 + 100(200) = \$25\,000$

With discrimination, total revenue is

$100 \times 200 + 100 \times 150 = \$35\,000$

so profit is \$35 000 – \$25 000 = \$10 000.

Without discrimination, total revenue is

$$200 \times \frac{500}{3} = \$33\,333$$

so profit is \$33 333 – \$25 000 = \$8333.

3 *Domestic market* From Practice Problem 2, profit is maximum when $P_1 = 200$, $Q_1 = 100$. Also, since $Q_1 = 300 - P_1$ we have $dQ_1/dP_1 = -1$. Hence

$$E_1 = -\frac{P_1}{Q_1} \times \frac{dQ_1}{dP_1} = -\frac{200}{100} \times (-1) = 2$$

Foreign market From Practice Problem 2, profit is maximum when $P_2 = 150$, $Q_2 = 100$. Also, since $Q_2 = 400 - 2P_2$ we have $dQ_2/dP_2 = -2$. Hence

$$E_2 = \frac{P_2}{Q_2} \times \frac{dQ_2}{dP_2} = -\frac{150}{100} \times (-2) = 3$$

We see that the firm charges the higher price in the domestic market, which has the lower elasticity of demand.

Exercise 4.7* (p. 355)

1 **(a)** $TR = aQ^2 + bQ$, $TC = dQ + c$

(b) $MR = 2aQ + b$, $MC = d$

(c) The equation $2aQ + b = d$ has solution

$$Q = \frac{d - b}{2a}$$

2 **(a)** At the point of maximum total revenue

$$MR = \frac{d(TR)}{dQ} = 0$$

so $E = 1$.

(b) Maximum occurs when $Q = 10$.

3 **(a)** $P_1 = \$30$, $P_2 = \$20$ **(b)** $P = \$24.44$

The profits in parts **(a)** and **(b)** are \$95 and \$83.89, respectively.

4 Argument is similar to that given in text but with $<$ replaced by $>$.

5 TC = ACO + ACC

$$= \frac{(ARU)(CO)}{EOQ} + (CU)(CC)\frac{(EOQ)}{2}$$

At a stationary point

$$\frac{d(TC)}{d(EOQ)} = -\frac{(ARU)(CO)}{(EOQ)^2} + \frac{(CU)(CC)}{2} = 0$$

which has solution

$$EOQ = \sqrt{\frac{2(ARU)(CO)}{(CU)(CC)}}$$

Also

$$\frac{d^2(TC)}{d(EOQ)^2} = \frac{2(ARU)(CO)}{(EOQ)^3} > 0$$

so minimum.

6 The argument is similar to that given in the text for AP_L.

7 The new supply equation is

$P = aQ_S + b + t$

In equilibrium

$aQ + b + t = -cQ + d$

which has solution

$$Q = \frac{d - b - t}{a + c}$$

Hence

$$tQ = \frac{(d - b)t - t^2}{a + c}$$

which differentiates to give

$$\frac{d - b - 2t}{a + c}$$

This is zero when

$$t = \frac{d - b}{2}$$

Also the second derivative is

$$\frac{-2}{a + c} < 0 \quad \text{(since } a \text{ and } c \text{ are both positive)}$$

which confirms that the stationary point is a maximum.

Section 4.8

Practice Problems

1

x	0.50	1.00	1.50	2.00
$f(x)$	−0.69	0.00	0.41	0.69
x	2.50	3.00	3.50	4.00
$f(x)$	0.92	1.10	1.25	1.39

The graph of the natural logarithm function is sketched in Figure S4.14.

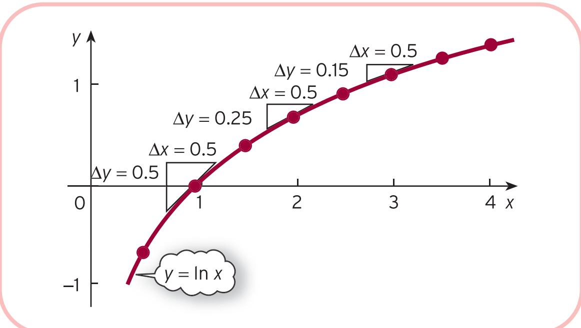


Figure S4.14

$$f'(1) = \frac{0.50}{0.50} = 1.00 = 1$$

$$f'(2) = \frac{0.25}{0.50} = 0.50 = \frac{1}{2}$$

$$f'(3) = \frac{0.15}{0.50} = 0.30 = \frac{1}{3}$$

These results suggest that $f'(x) = 1/x$.

2 **(a)** $3e^{3x}$ **(b)** $-e^{-x}$ **(c)** $1/x$ **(d)** $1/x$

3 **(a)** For the product rule we put

$$u = x^4 \quad \text{and} \quad v = \ln x$$

for which

$$\frac{du}{dx} = 4x^3 \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{x}$$

By the product rule

$$\frac{dy}{dx} = x^4 \times \frac{1}{x} + \ln x \times 4x^3$$

$$= x^3 + 4x^3 \ln x$$

$$= x^3 (1 + 4 \ln x)$$

(b) By the chain rule

$$\frac{dy}{dx} = e^{x^2} \times 2x = 2xe^{x^2}$$

(c) If

$$u = \ln x \quad \text{and} \quad v = x + 2$$

then

$$\frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dv}{dx} = 1$$

By the quotient rule

$$\frac{dy}{dx} = \frac{(x + 2) \times \frac{1}{x} - (\ln x) \times 1}{(x + 2)^2}$$

$$\frac{x + 2 - x \ln x}{x(x + 2)^2} \quad \text{(multiply top and bottom by } x\text{)}$$

4 **(a)** $y = \ln x^3 + \ln(x + 2)^4$ (rule 1)

$= 3 \ln x + 4 \ln(x + 2)$ (rule 3)

Hence

$$\frac{dy}{dx} = \frac{3}{x} + \frac{4}{x + 2}$$

$$\frac{3(x + 2) + 4x}{x(x + 2)} = \frac{7x + 6}{x(x + 2)}$$

(b) $y = \ln x^2 - \ln(2x + 3)$ (rule 2)

$= 2 \ln x - \ln(2x + 3)$ (rule 3)

Hence

$$\frac{dy}{dx} = \frac{2}{x} - \frac{2}{2x + 3} \quad \text{(chain rule)}$$

$$= \frac{2(2x + 3) - 2x}{x(2x + 3)}$$

$$= \frac{2x + 6}{x(2x + 3)}$$

5 In terms of P the total revenue function is given by

$$TR = PQ = 1000Pe^{-0.2P}$$

and the total cost function is

$$TC = 100 + 2Q = 100 + 2000e^{-0.2P}$$

Hence

$$\pi = TR - TC = 1000Pe^{-0.2P} - 2000e^{-0.2P} - 100$$

Step 1

At a stationary point

$$\frac{d\pi}{dP} = 0$$

To differentiate the first term, $1000Pe^{-0.2P}$, we use the product rule with

$$u = 1000P \quad \text{and} \quad v = e^{-0.2P}$$

for which

$$\frac{du}{dP} = 1000 \quad \text{and} \quad \frac{dv}{dP} = -0.2e^{-0.2P}$$

Hence the derivative of $1000Pe^{-0.2P}$ is

$$u\frac{dv}{dP} + v\frac{du}{dP} = 1000P(-0.2e^{-0.2P}) + e^{-0.2P}(1000)$$
$$= e^{-0.2P}(1000 - 200P)$$

Now

$$\pi = 1000Pe^{-0.2P} - 2000e^{-0.2P} - 100$$

so

$$\frac{d\pi}{dP} = e^{-0.2P}(1000 - 2000P) - 2000(-0.2e^{-0.2P})$$
$$= e^{-0.2P}(1400 - 200P)$$

This is zero when

$$1400 - 200P = 0$$

because $e^{-0.2P} \neq 0$.

Hence $P = 7$.

Step 2

To find $\frac{d^2\pi}{dP^2}$ we differentiate

$$\frac{d\pi}{dP} = e^{-0.2P}(1400 - 200P)$$

using the product rule. Taking

$$u = e^{-0.2P} \quad \text{and} \quad v = 1400 - 200P$$

gives

$$\frac{du}{dP} = -0.2e^{-0.2P} \quad \text{and} \quad \frac{dv}{dP} = -200$$

Hence

$$\frac{d^2\pi}{dP^2} = u\frac{dv}{dP} + v\frac{du}{dP}$$
$$= e^{-0.2P}(-200) + (1400 - 200P)(-0.2e^{-0.2P})$$
$$= e^{-0.2P}(10P - 480)$$

Putting $P = 7$ gives

$$\frac{d^2\pi}{dP^2} = -200e^{-1.4}$$

This is negative, so the stationary point is a maximum.

6 To find the price elasticity of demand we need to calculate the values of P, Q and dQ/dP. We are given that $Q = 20$ and the demand equation gives

$$P = 200 - 40 \ln(20 + 1) = 78.22$$

The demand equation expresses P in terms of Q, so we first evaluate dP/dQ and then use the result

$$\frac{dQ}{dP} = \frac{1}{dP/dQ}$$

To differentiate $\ln(Q + 1)$ by the chain rule we differentiate the outer log function to get

$$\frac{1}{Q + 1}$$

and then multiply by the derivative of the inner function, $Q + 1$, to get 1. Hence the derivative of $\ln(Q + 1)$ is

$$\frac{1}{Q + 1}$$

and so

$$\frac{dP}{dQ} = \frac{-40}{Q + 1}$$

Putting $Q = 20$ gives $dP/dQ = -40/21$, so that $dQ/dP = -21/40$. Finally, we use the formula

$$E = -\frac{P}{Q} \times \frac{dQ}{dP}$$

to calculate the price elasticity of demand as

$$E = -\frac{78.22}{20} \times \left(\frac{-21}{40}\right) = 2.05$$

Exercise 4.8 (p. 365)

1 **(a)** $6e^{6x}$ **(b)** $-342e^{-342x}$

(c) $-2e^{-x} + 4e^{x}$ **(d)** $40e^{4x} - 4x$

2 **(1)** **(a)** \$4885.61 **(b)** \$4887.57; 196

(2) $160e^{0.04t}$; 195.42

3 **(a)** $\frac{1}{x}$ **(b)** $\frac{1}{x}$

4 **(a)** $3x^2e^{x^3}$ **(b)** $\frac{4x^3 + 6x}{x^4 + 3x^2}$

5 **(a)** $(4x^3 + 2x^4)e^{2x}$ **(b)** $\ln x + 1$

6 **(a)** $\frac{2e^{4x}(2x^2 - x + 4)}{(x^2 + 2)^2}$ **(b)** $\frac{e^x(x \ln x - 1)}{x(\ln x)^2}$

7 (a) Maximum at $(1, e^{-1})$; the graph is sketched in Figure S4.15.

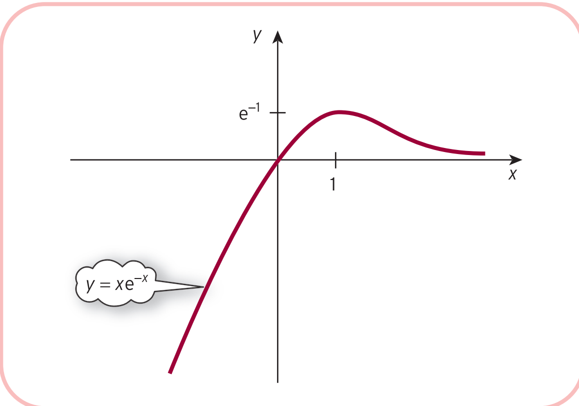


Figure S4.15

(b) Maximum at $(1, -1)$; the graph is sketched in Figure S4.16.

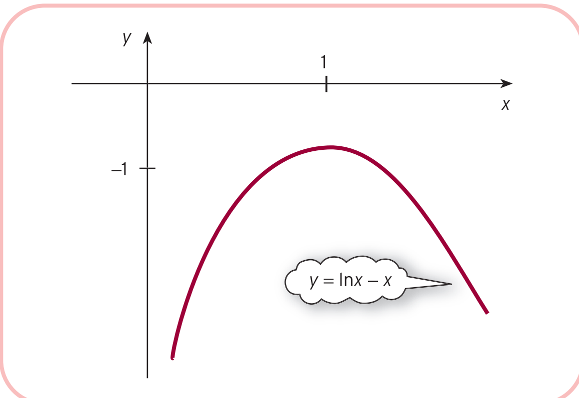


Figure S4.16

8 49

9 50

10 $E = \frac{10}{Q}$, which is 1 when $Q = 10$.

Exercise 4.8* (p. 366)

1 (a) $2e^{2x} + 12e^{-4x}$ (b) $(4x + 1)e^{4x}$

(c) $\frac{-(x+2)e^{-x}}{x^3}$ (d) $(m \ln x + 1)x^{m-1}$ (e) $\ln x$

(f) $\frac{(n \ln x - 1)x^{n-1}}{(\ln x)^2}$ (g) $\frac{(amx + bm - an)e^{mx}}{(ax+b)^{n+1}}$

(h) $\frac{(ax \ln bx - n)e^{ax}}{x(\ln bx)^{n+1}}$ (i) $\frac{2e^x}{(e^x + 1)^2}$

2 (a) $\frac{1}{x(1+x)}$ (b) $\frac{9x-2}{2x(3x-1)}$ (c) $\frac{1}{1-x^2}$

3 $\frac{dy}{dx} \div y = Ake^{kt} \div Ae^{kt} = k$

4 (a) $4x^3(1 - x^2)e^{-2x^2}$ (b) $\frac{1-x}{x(x+1)}$

5 (a) maximum at $\left(\frac{-1}{a}, \frac{-e^{-1}}{a}\right)$

(b) maximum at $\left(\frac{-b}{2a}, \ln\left(\frac{-b^2}{4a}\right)\right)$

6 (b) $\frac{4(x+1)}{(2x+1)(4x+3)}$

7 100

8 (a) $\frac{1}{Q}$ (b) $\frac{(2Q+1)(20 - 3\ln(2Q+1))}{6Q}$

9 (a) $\frac{(2Q^2 - 1)e^{Q^2}}{Q^2}$ (b) $\frac{1}{3Q+1} + \ln\left(\frac{2Q}{3Q+1}\right)$

Chapter 5

Section 5.1

Practice Problems

11 (a) −10 (b) −1 (c) 2 (d) 21 (e) 0

(f) 21. The value of g is independent of the ordering of the variables. Such a function is said to be *symmetric*.

2 (a) Differentiating $5x^4$ with respect to x gives $20x^3$ and, since y is held constant, y^2 differentiates to zero.

Hence

$$\frac{\partial f}{\partial x} = 20x^3 - 0 = 20x^3$$

Differentiating $5x^4$ with respect to y gives zero because x is held fixed. Also differentiating y^2 with respect to y gives $2y$, so

$$\frac{\partial f}{\partial y} = 0 - 2y = -2y$$

(b) To differentiate the first term, x^2y^3, with respect to x we regard it as a constant multiple of x^2 (where the constant is y^3), so we get $2xy^3$. The second term obviously gives −10, so

$$\frac{\partial f}{\partial x} = 2xy^3 - 10$$

To differentiate the first term, x^2y^3, with respect to y we regard it as a constant multiple of y^3

(where the constant is x^2), so we get $3x^2y^2$. The second term is a constant and goes to zero, so

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 0 = 3x^2y^2$$

3 (a) $f_{xx} = 60x^2$

$f_{yy} = 2$

$f_{yx} = f_{xy} = 0$

(b) $f_{xx} = 2y^3$

$f_{yy} = 6x^2y$

$f_{yx} = f_{xy} = 6xy^2$

4 $f_1 = \dfrac{\partial f}{\partial x_1} = x_2 + 5x_1^4$

$$f_{11} = \frac{\partial^2 f}{\partial x_1^2} = 20x_1^3$$

$$f_{21} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

5 $\dfrac{\partial z}{\partial x} = y - 5$, $\dfrac{\partial z}{\partial y} = x + 2$, so, at (2, 6),

$$\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = 4$$

(a) $\Delta x = -0.1$, $\Delta y = 0.1$;

$z = 1(-0.1) + 4(0.1) = 0.3$, so z increases by approximately 0.3.

(b) At (2, 6), $z = 14$, and at (1.9, 6.1), $z = 14.29$, so the exact increase is 0.29.

6 (a) $\dfrac{dy}{dx} = \dfrac{-y}{x - 3y^2 + 1}$

(b) $\dfrac{dy}{dx} = \dfrac{y^2}{5y^4 - 2xy}$

Exercise 5.1 (p. 382)

1 324; 75; 0

2 (a) $f_x = 2x$, $f_y = 20y^4$

(b) $f_x = 9x^2$, $f_y = -2e^y$

(c) $f_x = y$, $f_y = x + 6$

(d) $f_x = 6x^5y^2$, $f_y = 2x^6y + 15y^2$

3 $f_x = 4x^3y^5 - 2x$

$f_y = 5x^4y^4 + 2y$

$f_x(1, 0) = -2$

$f_y(1, 1) = 7$

4 (a) −0.6 (b) −2 (c) −2.6

5 (a) $f_x = -3x^2 + 2$, $f_y = 1$

$$\frac{dy}{dx} = \frac{-3x^2 + 2}{1} = 3x^2 - 2$$

(b) $y = x^3 - 2x + 1$, so

$$\frac{dy}{dx} = 3x^2 - 2 \quad ✓$$

Exercise 5.1* (p. 383)

1 $85 \neq 91$; $(0, y)$ for any y.

2

	f_x	f_y	f_{xx}	f_{yy}	f_{yx}	f_{xy}
(a)	Y	x	0	0	1	1
(b)	e^xy	e^x	e^xy	0	e^x	e^x
(c)	$2x + 2$	1	2	0	0	0
(d)	$4x^{-3/4}y^{3/4}$	$12x^{1/4}y^{-1/4}$	$-3x^{-7/4}y^{3/4}$	$-3x^{1/4}y^{-5/4}$	$3x^{-3/4}y^{-1/4}$	$3x^{-3/4}y^{-1/4}$
(e)	$\frac{-2y}{x^3} + \frac{1}{y}$	$\frac{1}{x^2} - \frac{x}{y^2}$	$\frac{6y}{x^4}$	$\frac{2x}{y^2}$	$\frac{-2}{x^3} - \frac{1}{y^2}$	$\frac{-2}{x^3} - \frac{1}{y^2}$

3 78; 94; 6.2

4 1/3

5 $f_1 = \dfrac{x_3^2}{x_2}$; $f_2 = -\dfrac{x_1x_3^2}{x_2^2} + \dfrac{1}{x_2}$; $f_3 = \dfrac{2x_1x_3}{x_2} + \dfrac{1}{x_3}$;

$$f_{11} = 0;\ f_{22} = \frac{2x_1x_3^2}{x_2^3} - \frac{1}{x_2^2};\ f_{33} = \frac{2x_1}{x_2} - \frac{1}{x_3^2};$$

$$f_{12} = -\frac{x_3^2}{x_2^2} = f_{21};\ f_{13} = -\frac{2x_3}{x_2} = f_{31};$$

$$f_{23} = \frac{-2x_1x_3}{x_2^2} = f_{32}$$

6 e.g. $f(x, y) = x^3y^2 + 3x^2y$

7 5

8 2.5

Section 5.2

Practice Problems

1 Substituting the given values of P, P_A and Y into the demand equation gives

$$Q = 500 - 3(20) - 2(30) + 0.01(5000) = 430$$

(a) $\dfrac{\partial Q}{\partial P} = -3$, so

$$E_P = \frac{-20}{430} \times (-3) = 0.14$$

(b) $\dfrac{\partial Q}{\partial P_A} = -2$, so

$$E_{P_A} = \frac{30}{430} \times (-2) = 0.14$$

(c) $\dfrac{\partial Q}{\partial Y} = 0.01$, so

$$E_Y = \frac{5000}{430} \times 0.01 = 0.12$$

By definition,

$$E_Y = \frac{\text{percentage change in } Q}{\text{percentage change in } Y}$$

so demand rises by $0.12 \times 5 = 0.6\%$. A rise in income causes a rise in demand, so good is superior.

2 $\frac{\partial U}{\partial x_1} = 1000 + 5x_2 - 4x_1$

$\frac{\partial U}{\partial x_2} = 450 + 5x_1 - 2x_2$, so at (138, 500)

$\frac{\partial U}{\partial x_1} = 2948$ and $\frac{\partial U}{\partial x_2} = 140$

If working time increases by 1 hour then leisure time decreases by 1 hour, so $\Delta x_1 = -1$. Also $\Delta x_2 = 15$. By the small increments formula

$$\Delta U \simeq 2948(-1) + 140(15) = -848$$

The law of diminishing marginal utility holds for both x_1 and x_2 because

$$\frac{\partial^2 U}{\partial x_1^2} = -4 < 0$$

and

$$\frac{\partial^2 U}{\partial x_2^2} = -2 < 0$$

3 Using the numerical results in Practice Problem 2,

$$\text{MRCS} = \frac{2948}{140} = 21.06$$

This represents the increase in x_2 required to maintain the current level of utility when x_1 falls by 1 unit. Hence if x_1 falls by 2 units, the increase in x_2 is approximately

$$21.06 \times 2 = \$42.12$$

4 $MP_K = 2K$ and $MP_L = 4L$

(a) $\text{MRTS} = \frac{MP_L}{MP_K} = \frac{4L}{2K} = \frac{2L}{K}$

(b) $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K(2K) + L(4L)$

$$= 2(K^2 + 2L^2) = 2Q \quad ✓$$

Exercise 5.2 (p. 398)

1 2.71

2 **(a)** 20/1165 **(b)** −15/1165

(c) 2000/1165; −0.04%; complementary

3 2; 1%

4 $\frac{\partial U}{\partial x_1} = \frac{1}{5}$ and $\frac{\partial U}{\partial x_2} = \frac{5}{12}$

(a) 37/60 **(b)** 12/25

5 $MP_K = 8$, $MP_L = 14^1/_4$ **(a)** $1^{25}/_{32}$ **(b)** $1^{25}/_{32}$

6 $K(6K^2 + 3L^2) + L(6LK) = 6K^3 + 9L^2K = 3(K^3 + 3L^2K)$

7 **(b)** $\frac{0.4(5\sqrt{K} + 1.5\sqrt{L})\sqrt{K}}{L}$

(c) 2.97

Exercise 5.2* (p. 399)

1 **(a)** 0.4 **(b)** 3.2%

Since $E < 1$, the good is income-inelastic. The relative market share of the good decreases as the economy expands.

2 0.5

3 16

4 $\frac{\partial Q}{\partial K} = \alpha AK^{\alpha-1}L^{\beta}$ and

$\frac{\partial Q}{\partial L} = \beta AK^{\alpha}L^{\beta-1}$, so

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = \alpha AK^{\alpha}L^{\beta} + \beta AK^{\alpha}L^{\beta} = (\alpha + \beta)(AK^{\alpha}L^{\beta})$$

$$= (\alpha + \beta)Q \quad ✓$$

5 The graph is sketched in Figure S5.1, which shows that MRTS = −(−5/7) = 5/7.

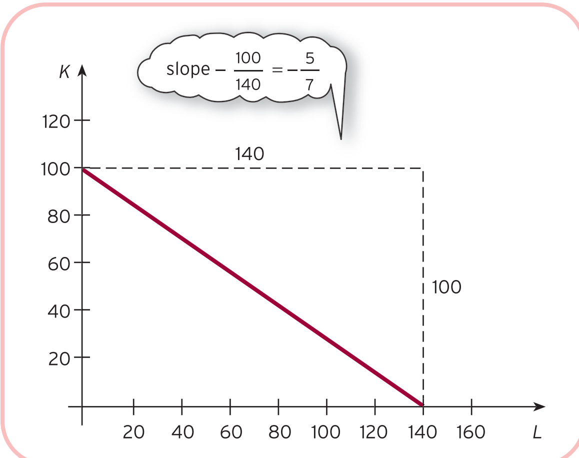


Figure S5.1

6 **(a)** $MP_K = \frac{10}{3}$; $MP_L = \frac{21}{2}$

(b) $\Delta Q \approx -11$

(c) 3.15; e.g. a 1 unit increase in labour and a 3.15 decrease in capital maintains a constant level of output.

7 **(a)** $MP_K = AbK^{\alpha-1}[\quad]^{\frac{1}{\alpha}-1}$; $MP_L = A(1-b)L^{\alpha-1}[\quad]^{\frac{1}{\alpha}-1}$

$$\text{MRTS} = \frac{MP_L}{MP_K} = \frac{1-b}{b} \times \frac{L^{\alpha-1}}{K^{\alpha-1}} = \frac{1-b}{b}\left(\frac{K}{L}\right)^{1-\alpha}$$

(b) $KAbK^{\alpha-1}[\quad]^{\frac{1}{\alpha}-1} + LA(1-b)L^{\alpha-1}[\quad]^{\frac{1}{\alpha}-1}$

$= A(bK^{\alpha} + (1-b)L^{\alpha})[\quad]^{\frac{1}{\alpha}-1}$

$= A[\quad]^{1/\alpha}$

$= Q$

8 (a) Complementary; coefficient of P_A is negative.

(b) (i) $E_P = \dfrac{b}{100}$

(ii) $E_{P_A} = -\dfrac{3c}{500}$

(iii) $E_Y = \dfrac{d}{5}$

(c) $a = 4310, b = 5, c = 2, d = 1$

9 (c) $\dfrac{7K^4}{3L^4}$ **(d)** -0.033

Section 5.3

Practice Problems

1 $C = a\left(\dfrac{b + I^*}{I - a}\right) + b$

$$\frac{\partial S}{\partial I^*} = \frac{a}{1-a} > 0$$

because

$0 < a < 1$

Hence an increase in I^* leads to an increase in C.
If $a = {}^1/_2$ then

$$\frac{\partial C}{\partial I^*} = \frac{{}^1/_2}{1 - {}^1/_2} = 1$$

Change in C is

$1 \times 2 = 2$

2 (a) Substitute C, I, G, X and M into the Y equation to get

$$Y = aY + b + I^* + G^* + X^* - (mY + M^*)$$

Collecting like terms gives

$$(1 - a + m)Y = b + I^* + G^* + X^* - M^*$$

so

$$Y = \frac{b + I^* + G^* + X^* - M^*}{1 - a + m}$$

(b) $\dfrac{\partial Y}{\partial X^*} = \dfrac{1}{1 - a + m}$

$$\frac{\partial Y}{\partial m} = -\frac{b + I^* + G^* + X^* - M^*}{(1 - a + m)^2}$$

Now $a < 1$ and $m > 0$, so $1 - a + m > 0$. The autonomous export multiplier is positive, so an increase in X^* leads to an increase in Y. The marginal propensity to import multiplier is negative. To see this note from part **(a)** that $\partial Y/\partial m$ can be written as

$$\frac{Y}{1 - a + m}$$

and $Y > 0$ and $1 - a + m > 0$.

(c) $Y = \dfrac{120 + 100 + 300 + 150 - 40}{1 - 0.8 + 0.1}$

$= 2100$

$$\frac{\partial Y}{\partial X^*} = \frac{1}{1 - 0.8 + 0.1} = \frac{10}{3}$$

and

$\Delta X^* = 10$

so

$$\Delta Y = \frac{10}{3} \times 10 = \frac{100}{3}$$

3 If d increases by a small amount then the intercept increases and the demand curve shifts upwards slightly. Figure S5.2 shows that the effect is to increase the equilibrium quantity from Q_1 to Q_2, confirming that $\partial Q/\partial d > 0$.

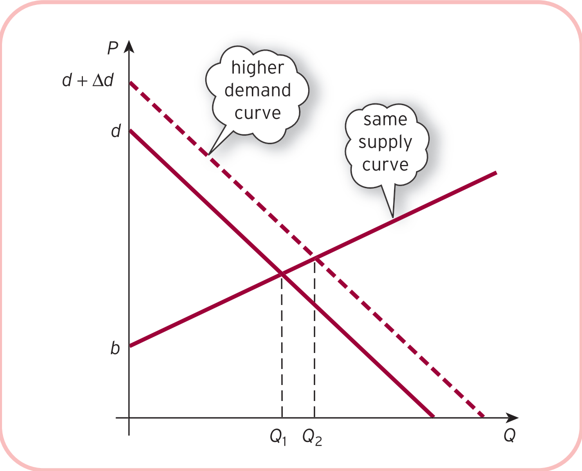


Figure S5.2

Exercise 5.3* (p. 412)

1 (a) Substituting equations (3) and (4) into (2) gives

$$C = a(Y - T^*) + b = aY - aT^* + b \qquad (7)$$

Substituting (5), (6) and (7) into (1) gives

$$Y = aY - aT^* + b + I^* + G^*$$

so that

$$Y = \frac{-aT^* + b + I^* + G^*}{1 - a}$$

Finally, from (7), we see that

$$C = a\left(\frac{-aT^* + b + I^* + G^*}{1-a}\right) - aT^* + b$$

$$= \frac{a(-aT^* + b + I^* + G^*) + (1-a)(-aT^* + b)}{1-a}$$

$$= \frac{aI^* + aG^* - aT^* + b}{1-a}$$

(b) $\frac{a}{1-a} > 0$; C increases. **(c)** 1520; rise of 18.

2 $\frac{-a(b + I^* + G^* - aT^*)}{(1 - a - at)^2}$

3 **(1)** From the relations

$C = aY_d + b$

$Y_d = Y - T$

$T = tY + T^*$

we see that

$C = a(Y - tY - T^*) + b$

Similarly,

$M = m(Y - tY - T^*) + M^*$

Substitute these together with I, G and X into the Y equation to get the desired result.

(2) **(a)** $\frac{\partial Y}{\partial T^*} = \frac{m - a}{1 - a + at + m - mt}$

Numerator is negative because $m < a$. Denominator can be written as

$(1 - a) + at + m(1 - t)$

which represents the sum of three positive numbers, so is positive. Hence the autonomous taxation multiplier is negative.

(b) $\frac{\partial Y}{\partial G^*} = \frac{1}{1 - a + at + m - mt} > 0$

(3) **(a)** 1000 **(b)** $\Delta Y = 20$ **(c)** $\Delta T^* = 33^1/_3$

4 From text, equilibrium quantity is

$$\frac{d-b}{a+c}$$

Substituting this into either the supply or demand equation gives the desired result:

$$\frac{\partial P}{\partial a} = \frac{c(d-b)}{(a+c)^2} > 0, \quad \frac{\partial P}{\partial b} = \frac{c}{a+c} > 0$$

$$\frac{\partial P}{\partial c} = -\frac{a(d-b)}{(a+c)^2} < 0, \quad \frac{\partial P}{\partial d} = \frac{a}{a+c} > 0$$

where the quotient rule is used to obtain $\partial P/\partial a$ and $\partial P/\partial c$. An increase in a, b or d leads to an increase in P, whereas an increase in c leads to a decrease in P.

5 **(a)** $C = aY_d + b = a(Y - T) + b = a(Y - T^*) + b$

$Y = C + I + G = a(Y - T^*) + b + I^* + G^*$

$(1 - a)Y = -aT^* + b + I^* + G^* \Rightarrow$

$Y = \frac{1}{1-a}(b - aT^* + I^* + G^*)$

(b) $\frac{\partial Y}{\partial G^*} = \frac{1}{1-a}$; $\frac{\partial Y}{\partial T^*} = -\frac{a}{1-a}$; $\Delta Y = 1$

(c) 1

6 **(1)** Substituting second and third equations into first gives

$Y = aY + b + cr + d$

so that

$(1 - a)Y - cr = b + d$ (1)

(2) Substituting first and second equations into third gives

$k_1Y + k_2r + k_3 = M_S^*$

so that

$k_1Y + k_2r = M_S^* - k_3$ (2)

(3) **(a)** Working out $c \times (2) + k_2 \times (1)$ eliminates r to give

$ck_1Y + k_2(1 - a)Y = c(M_S^* - k_3) + k_2(b + d)$

Dividing both sides by $ck_1 + k_2(1 - a)$ gives result.

(b) $\frac{c}{(1-a)k_2 + ck_1}$, which is positive because the top and bottom of this fraction are both negative.

7 **(a)** $Y_d = Y - T = Y - (tY + T^*) = (1 - t)Y - T^*$

$C = aY_d + b = a[(1 - t)Y - T^*] + b = a(1 - t)Y - aT^* + b$

$Y = C + I = a(1 - t)Y - aT^* + b + cr + d$

$(1 - a(1 - t))Y = -aT^* + b + cr + d \Rightarrow$

$Y = \frac{b + d - aT^* + cr}{1 - a(1-t)}$

(b) $\frac{\partial Y}{\partial c} = \frac{r}{1 - a(1-t)}$

$\frac{\partial Y}{\partial a} = \frac{-T^* + (1-t)(b + d + cr)}{(1 - a(1-t))^2}$

(c) $0 < t < 1$ and $0 < a < 1 \Rightarrow 0 < a(1 - t) < 1$

$\Rightarrow 1 - a(1 - t) > 0$

$r > 0 \Rightarrow \frac{\partial Y}{\partial c} > 0 \Rightarrow Y$ increases

(d) $Y = 2800$; $\Delta Y = 46.25$

Section 5.4

Practice Problems

1 $f_x = 2x$, $f_y = 6 - 6y$, $f_{xx} = 2$, $f_{yy} = -6$, $f_{xy} = 0$

Step 1

At a stationary point

$$2x = 0$$
$$6 - 6y = 0$$

which shows that there is just one stationary point at (0, 1).

Step 2

$$f_{xx}f_{yy} - f_{xy}^2 = 2(-6) - 0^2 = -12 < 0$$

so it is a saddle point.

2 Total revenue from the sale of G1 is

$$TR_1 = P_1Q_1 = (50 - Q_1)Q_1 = 50Q_1 - Q_1^2$$

Total revenue from the sale of G2 is

$$TR_2 = P_2Q_2 = (95 - 3Q_2)Q_2 = 95Q_2 - 3Q_2^2$$

Total revenue from the sale of both goods is

$$TR = TR_1 + TR_2 = 50Q_1 - Q_1^2 + 95Q_2 - 3Q_2^2$$

Profit is

$$\pi = TR - TC = (50Q_1 - Q_1^2 + 95Q_2 - 3Q_2^2) - (Q_1^2 + 3Q_1Q_2 + Q_2^2) = 50Q_1 - 2Q_1^2 + 95Q_2 - 4Q_2^2 - 3Q_1Q_2$$

Now

$$\frac{\partial\pi}{\partial Q_1} = 50 - 4Q_1 - 3Q_2,$$

$$\frac{\partial\pi}{\partial Q_2} = 95 - 8Q_2 - 3Q_1$$

$$\frac{\partial^2\pi}{\partial Q_1^2} = -4, \quad \frac{\partial^2\pi}{\partial Q_1\partial Q_2} = -3,$$

$$\frac{\partial^2\pi}{\partial Q_2^2} = -8$$

Step 1

At a stationary point

$$50 - 4Q_1 - 3Q_2 = 0$$
$$95 - 3Q_1 - 8Q_2 = 0$$

that is,

$$4Q_1 + 3Q_2 = 50 \qquad (1)$$
$$3Q_1 + 8Q_2 = 95 \qquad (2)$$

Multiply equation (1) by 3, and equation (2) by 4 and subtract to get

$$23Q_2 = 230$$

so $Q_2 = 10$. Substituting this into either equation (1) or equation (2) gives $Q_1 = 5$.

Step 2

This is a maximum because

$$\frac{\partial^2\pi}{\partial Q_1^2} = -4 < 0, \quad \frac{\partial^2\pi}{\partial Q_2^2} = -8 < 0$$

and

$$\left(\frac{\partial^2\pi}{\partial Q_1^2}\right)\left(\frac{\partial^2\pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2\pi}{\partial Q_1\partial Q_2}\right)^2 = (-4)(-8) - (-3)^2 = 23 > 0$$

Corresponding prices are found by substituting $Q_1 = 5$ and $Q_2 = 10$ into the original demand equations to obtain $P_1 = 45$ and $P_2 = 65$.

3 For the domestic market, $P_1 = 300 - Q_1$, so

$$TR_1 = P_1Q_1 = 300Q_1 - Q_1^2$$

For the foreign market, $P_2 = 200 - {}^1\!/_2Q_2$, so

$$TR_2 = P_2Q_2 = 200Q_2 - {}^1\!/_2Q_2^2$$

Hence

$$TR = TR_1 + TR_2 = 300Q_1 - Q_1^2 + 200Q_2 - {}^1\!/_2Q_2^2$$

We are given that

$$TC = 5000 + 100(Q_1 + Q_2) = 5000 + 100Q_1 + 100Q_2$$

so

$$\pi = TR - TC = (300Q_1 - Q_1^2 + 200Q_2 - {}^1\!/_2Q_2^2) - (5000 + 100Q_1 + 100Q_2) = 200Q_1 - Q_1^2 + 100Q_2 - {}^1\!/_2Q_2^2 - 5000$$

Now

$$\frac{\partial\pi}{\partial Q_1} = 200 - 2Q_1, \quad \frac{\partial\pi}{\partial Q_2} = 100 - 2Q_2$$

$$\frac{\partial^2\pi}{\partial Q_1^2} = -2, \quad \frac{\partial^2\pi}{\partial Q_1^2\partial Q_2} = 0, \quad \frac{\partial^2\pi}{\partial Q_2^2} = -1$$

Step 1

At a stationary point

$$200 - 2Q_1 = 0$$
$$100 - Q_2 = 0$$

which have solution $Q_1 = 100$, $Q_2 = 100$.

Step 2

This is a maximum because

$$\frac{\partial^2\pi}{\partial Q_1^2} = -2 < 0,$$

$$\frac{\partial^2\pi}{\partial Q_2^2} = -1 < 0$$

and

$$\left(\frac{\partial^2\pi}{\partial Q_1^2}\right)\left(\frac{\partial^2\pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2\pi}{\partial Q_1\partial Q_2}\right)^2 = (-2)(-1) - 0^2 = 2 > 0$$

Substitute $Q_1 = 100$, $Q_2 = 100$, into the demand and profit functions to get $P_1 = 200$, $P_2 = 150$ and $\pi = 10\,000$.

Exercise 5.4 (p. 427)

1 (a) Minimum at (1, 1), maximum at (−1, −1), and saddle points at (1, −1) and (−1, 1).

(b) Minimum at (2, 0), maximum at (0, 0), and saddle points at (1, 1) and (1, −1).

2 $Q_1 = 9, \qquad Q_2 = 6$

$$\frac{\partial^2\pi}{\partial Q_1^2} = -2 < 0, \ \frac{\partial^2\pi}{\partial Q_2^2} = -4 < 0$$

$$\left(\frac{\partial^2\pi}{\partial Q_1^2}\right)\left(\frac{\partial^2\pi}{\partial Q_2^2}\right) - \left(\frac{\partial^2\pi}{\partial Q_1 \partial Q_2}\right)^2$$

$$= (-2)(-4) - (-1)^2 = 7 > 0 \quad \Rightarrow \quad \text{max}$$

3 Maximum profit is \$1300 when $Q_1 = 30$ and $Q_2 = 10$.

4 $x_1 = 138$, $x_2 = 500$; \$16.67 per hour.

5 $Q_1 = 19$, $Q_2 = 4$

6 (a) Minimum at (1, 2).

(b) Maximum at (0, 1).

(c) Saddle point at (2, 2).

Exercise 5.4* (p. 428)

1 Minimum at (0, 0) and saddle point at (−0.5, −0.25).

2 Maximum profit is \$176 when $L = 16$ and $K = 144$.

3 \$346 500

4 Maximum profit is \$95 when $P_1 = 30$ and $P_2 = 20$.

5 (a) $P_1 = 55 \quad P_2 = 0.5a + 5$

(b) $P = 15 + 0.4a$

Profit under **(a)** is $\pi_a = 512.5 + 0.5(a - 10)^2$

Profit under **(b)** is $\pi_b = -437.5 + 10a + 0.4a^2$

$\pi_a - \pi_b = 0.1(a - 100)^2 \geq 0$

6 (a) $P_1 = 78$, $P_2 = 68$

(b) Rotate the box so that the Q_1 axis comes straight out of the screen. The graph increases steadily as Q_2 rises from 0 to 2.

$Q_1 = 24$. Profit in **(a)** and **(b)** is 1340 and 1300 respectively.

Section 5.5

Practice Problems

1 **Step 1**

We are given that $y = x$, so no rearrangement is necessary.

Step 2

Substituting $y = x$ into the objective function

$$z = 2x^2 - 3xy + 2y + 10$$

gives

$$z = 2x^2 - 3x^2 + 2x + 10 = -x^2 + 2x + 10$$

Step 3

At a stationary point

$$\frac{dz}{dx} = 0$$

that is,

$$-2x + 2 = 0$$

which has solution $x = 1$. Differentiating a second time gives

$$\frac{d^2z}{dx^2} = -2$$

confirming that the stationary point is a maximum.

The value of z can be found by substituting $x = 1$ into

$$z = -2x^2 + 2x + 10$$

to get $z = 11$. Finally, putting $x = 1$ into the constraint $y = x$ gives $y = 1$. The constrained function therefore has a maximum value of 11 at the point (1, 1).

2 We want to maximize the objective function

$$U = x_1x_2$$

subject to the budgetary constraint

$$2x_1 + 10x_2 = 400$$

Step 1

$$x_1 = 200 - 5x_2$$

Step 2

$$U = 200x_2 - 5x_2^2$$

Step 3

$$\frac{dU}{dx_1} = 200 - 10x_2 = 0$$

has solution $x_2 = 20$.

$$\frac{d^2U}{dx_2^2} = -10 < 0$$

so maximum.

Putting $x_2 = 20$ into constraint gives $x_1 = 100$.

$$U_1 = \frac{\partial U}{\partial x_1} = x_2 = 20$$

and

$$U_2 = \frac{\partial U}{\partial x_2} = x_2 = 100$$

so the ratios of marginal utilities to prices are

$$\frac{U_1}{P_1} = \frac{20}{2} = 10$$

and

$$\frac{U_2}{P_2} = \frac{100}{10} = 10$$

which are the same.

3 We want to minimize the objective function

$$TC = 3x_1^2 + 2x_1x_2 + 7x_2^2$$

subject to the production constraint

$$x_1 + x_2 = 40$$

Step 1

$$x_1 = 40 - x_2$$

Step 2

$$TC = 3(40 - x_2)^2 + 2(40 - x_2)x_2 + 7x_2^2$$
$$= 4800 - 160x_2 + 8x_2^2$$

Step 3

$$\frac{d(TC)}{dx_2} = -160 + 16x_2 = 0$$

has solution $x_2 = 10$.

$$\frac{d^2(TC)}{dx_2^2} = 16 > 0$$

so minimum.

Finally, putting $x_2 = 10$ into constraint gives $x_1 = 30$.

Exercise 5.5 (p. 440)

1 **(a)** $y = \frac{2}{3} - 3x$

(b) Maximum value of z is $\frac{1}{9}$ which occurs at $\left(\frac{1}{9}, \frac{1}{3}\right)$.

2 Maximum value of z is 13, which occurs at (3, 11).

3 27 000.

4 $K = 6$ and $L = 4$.

5 $x + y = 20$; 6 of product A and 14 of product B

6 8100

Exercise 5.5* (p. 441)

1 **(a)** Minimum value of 1800 occurs at (10, 30).

(b) Maximum value of 10 occurs at $\left(\frac{1}{2}, 2\right)$.

2 $K = 10$ and $L = 4$.

3 Maximum profit is \$165, which is achieved when $K = 81$ and $L = 9$.

4 $x_1 = 3, x_2 = 4$

Section 5.6

Practice Problems

1 **Step 1**

$$g(x, y, \lambda) = 2x^2 - xy + \lambda(12 - x - y)$$

Step 2

$$\frac{\partial g}{\partial x} = 4x - y - \lambda = 0$$

$$\frac{\partial g}{\partial y} = -x - \lambda = 0$$

$$\frac{\partial g}{\partial \lambda} = 12 - x - y = 0$$

that is,

$$4x - y - \lambda = 0 \quad (1)$$
$$-x - \lambda = 0 \quad (2)$$
$$x + y = 12 \quad (3)$$

Multiply equation (2) by 4 and add equation (1), multiply equation (3) by 4 and subtract from equation (1) to get

$$-y - 5\lambda = 0 \quad (4)$$
$$-5y - \lambda = -48 \quad (5)$$

Multiply equation (4) by 5 and subtract equation (5) to get

$$-24\lambda = 48 \quad (6)$$

Equations (6), (5) and (1) can be solved in turn to get

$$\lambda = -2, y = 10, x = 2$$

so the optimal point has coordinates (2, 10). The corresponding value of the objective function is

$$2(2)^2 - 2(10) = -12$$

2 Maximize

$$U = 2x_1x_2 + 3x_1$$

subject to

$$x_1 + 2x_2 = 83$$

Step 1

$$g(x_1, x_2, \lambda) = 2x_1x_2 + 3x_1 + \lambda(83 - x_1 - 2x_2)$$

Step 2

$$\frac{\partial g}{\partial x_1} = 2x_2 + 3 - \lambda = 0$$

$$\frac{\partial g}{\partial x_2} = 2x_1 - 2\lambda = 0$$

$$\frac{\partial g}{\partial \lambda} = 83 - x_1 - 2x_2 = 0$$

that is,

$$2x_2 - \lambda = -3 \quad (1)$$
$$2x_1 - 2\lambda = 0 \quad (2)$$
$$x_1 + 2x_2 = 83 \quad (3)$$

The easiest way of solving this system is to use equations (1) and (2) to get

$\lambda = 2x_2 + 3$ and $\lambda = x_1$ respectively.

Hence

$x_1 = 2x_2 + 3$

Substituting this into equation (3) gives

$4x_2 + 3 = 83$

which has solution $x_2 = 20$ and so $x_1 = \lambda = 43$.

The corresponding value of U is

$2(43)(20) + 3(43) = 1849$

The value of λ is 43, so when income rises by 1 unit, utility increases by approximately 43 to 1892.

3 Step 1

$$g(x_1, x_2, \lambda) = x_1^{1/2} + x_2^{1/2} + \lambda(M - P_1x_1 - P_2x_2)$$

Step 2

$$\frac{\partial g}{\partial x_1} = \frac{1}{2}x_1^{-1/2} - \lambda P_1 = 0$$

$$\frac{\partial g}{\partial x_2} = \frac{1}{2}x_2^{-1/2} - \lambda P_2 = 0$$

$$\frac{\partial g}{\partial \lambda} = M - P_1x_1 - P_2x_2 = 0$$

From equations (1) and (2)

$$\lambda = \frac{1}{2x_1^{1/2}P_1} \quad \text{and} \quad \lambda = \frac{1}{2x_2^{1/2}P_2}$$

respectively. Hence

$$\frac{1}{2x_1^{1/2}P_1} = \frac{1}{2x_2^{1/2}P_2}$$

that is,

$$x_1P_1^2 = x_2P_2^2$$

so

$$x_1 = \frac{x_2P_2^2}{P_1^2} \qquad (4)$$

Substituting this into equation (3) gives

$$M - \frac{x_2P_2^2}{P_1} - P_2x_2 = 0$$

which rearranges as

$$x_2 = \frac{P_1M}{P_2(P_1 + P_2)}$$

Substitute this into equation (4) to get

$$x_1 = \frac{P_2M}{P_1(P_1 + P_2)}$$

Exercise 5.6 (p. 451)

1 9

2 **(a)** 800; $x = 20$, $y = 10$, $\lambda = 40$

(b) 840.5; $x = 20.5$, $y = 10.25$, $\lambda = 41$

(c) Change is 40.5 compared to a multiplier of 40.

3 4.5

4 Maximum profit is \$600 at $Q_1 = 10$, $Q_2 = 5$. Lagrange multiplier is 4, so profit rises to \$604 when total cost increases by 1 unit.

Exercise 5.6* (p. 452)

1 There are two wheels per frame, so the constraint is $y = 2x$. Maximum profit is \$4800 at $x = 40$, $y = 80$.

2 40; 2.5

3 $x_1 = \dfrac{\alpha M}{(\alpha + \beta)P_1}$ and $x_2 = \dfrac{\beta M}{(\alpha + \beta)P_2}$

4 $x = \$6715.56$; $y = \$3284.44$

5 $x = 13$, $y = 17$, $z = 2$

6 $x = 6$, $y = 11$

Chapter 6

Section 6.1

Practice Problems

1 **(a)** x^2 **(b)** x^4 **(c)** x^{100} **(d)** $\frac{1}{4}x^4$ **(e)** $\frac{1}{19}x^{19}$

2 **(a)** $\frac{1}{5}x^5 + c$ **(b)** $-\frac{1}{2x^2} + c$ **(c)** $\frac{3}{4}x^{4/3} + c$ **(d)** $\frac{1}{3}e^{3x} + c$

(e) $x + c$ **(f)** $\frac{x^2}{2} + c$ **(g)** $\ln x + c$

3 **(a)** $x^2 - x^4 + c$ **(b)** $2x^5 - \frac{5}{x} + c$

(c) $\frac{7}{3}x^3 - \frac{3}{2}x^2 + 2x + c$

4 **(a)** $\text{TC} = \int 2\,\text{d}Q = 2Q + c$

Fixed costs are 500, so $c = 500$. Hence

$\text{TC} = 2Q + 500$

Put $Q = 40$ to get $\text{TC} = 580$.

(b) $\text{TR} = \int (100 - 6Q)\text{d}Q = 100Q - 3Q^2 + c$

Revenue is zero when $Q = 0$, so $c = 0$. Hence

$\text{TR} = 100Q - 3Q^2$

$$P = \frac{\text{TR}}{Q} = \frac{100Q - 3Q^2}{Q} = 100 - 3Q$$

so demand equation is $P = 100 - 3Q$.

(c) $S = \int (0.4 - 0.1Y^{-1/2})\text{d}Y = 0.4Y - 0.2Y^{1/2} + c$

The condition $S = 0$ when $Y = 100$ gives

$0 = 0.4(100) - 0.2(100)^{1/2} + c = 38 + c$

so $c = -38$. Hence

$S = 0.4Y - 0.2Y^{1/2} - 38$

Exercise 6.1 (p. 468)

1 (a) $x^6 + c$ (b) $\frac{1}{5}x^5 + c$ (c) $e^{10x} + c$

(d) $\ln x + c$ (e) $\frac{2}{5}x^{5/2} + c$ (f) $\frac{1}{2}x^4 - 3x^2 + c$

(g) $\frac{1}{3}x^3 - 4x^2 + 3x + c$ (h) $\frac{ax^2}{2} + bx + c$

(i) $\frac{7}{4}x^4 - 2e^{-2x} + \frac{3}{x} + c$

2 (a) $TC = \frac{Q^2}{2} + 5Q + 20$ (b) $TC = 6e^{0.5Q} + 4$

3 (a) $TR = 20Q - Q^2$; $P = 20 - Q$

(b) $TR = 12\sqrt{Q}$; $P = \frac{12}{\sqrt{Q}}$

4 $C = 0.6Y + 7$, $S = 0.4Y - 7$

5 (a) $1000L - L^3$ (b) $12\sqrt{L} - 0.01L$

6 $\frac{Y}{3} + \sqrt{Y} + 3$; $k = 9$

7 (a) $500Le^{-0.02L}$ (b) $\ln(1 + 50Q^2)$

Exercise 6.1* (p. 469)

1 (a) $\frac{x^7}{7} - x^2 + c$

(b) $\frac{x^{11}}{11} - 2x\sqrt{x} - e^{-x} + c$

(c) $\frac{x^4}{4} + \frac{1}{x^5} + 2\ln x + e^{-4x} + c$

2 (a) $C = 20(Y + 2Y^{1/4} + 1)$ (b) $VC = 15 + Q^2$

3 (1) $F'(x) = 10(2x + 1)^4$, which is 10 times too big, so the integral is

$$\frac{1}{10}(2x + 1)^5 + c$$

(2) (a) $\frac{1}{24}(3x - 2)^8 + c$ (b) $-\frac{1}{40}(2 - 4x)^{10} + c$

(c) $\frac{1}{a(x + 1)}(ax + b)^{n+1} + c$ (d) $\frac{1}{7}\ln(7x + 3) + c$

4 (a) $\frac{1}{2}x^2 + \frac{2}{7}x^{7/2} + c$

(b) $\frac{1}{11}x^{11} + \frac{1}{3}x^3$; $\frac{1}{5}e^{5x} + e^x + \frac{3}{2}e^{2x} + c$; $\frac{1}{3}x^3 - \frac{1}{2}x^2 + c$

5 (a) $\frac{1}{4}x^4 - \frac{1}{2}x^2 + 2^{1/2}x + c$

(b) $\ln x + \frac{1}{x} + c$; $-e^{-x} + \frac{1}{3}e^{-3x} + c$; $x - x + \frac{2}{3}x^{3/2} + c$

6 $S = 0.6Y - 0.2\sqrt{Y} - 8$

7 550

Section 6.2

Practice Problems

1 (a) $\int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 = \frac{1}{4}$

(b) $\int_2^5 (2x - 1)dx = [x^2 - x]_2^5 = (5^2 - 5) - (2^2 - 2) = 18$

(c) $\int_1^4 (x^2 - x + 1)dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]_1^4$

$= \left[\frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 + 4\right] - \left[\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 + 1\right]$

$= 16.5$

(d) $\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1 = 1.718\ 28\ldots$

2 Substitute $Q = 8$ to get

$P = 100 - 8^2 = 36$

$CS = \int_0^8 (100 - Q^2)dQ - 8(36)$

$= \left[100Q - \frac{1}{3}Q^3\right]_0^8 - 288$

$= \left[100(8) - \frac{1}{3}(8)^3\right] - \left[100(0) - \frac{1}{3}(0)^2\right] - 288$

$= 341.33$

3 In equilibrium, $Q_S = Q_D = Q$, so

$P = 50 - 2Q$

$P = 10 + 2Q$

Hence

$50 - 2Q = 10 + 2Q$

which has solution $Q = 10$. The demand equation gives

$P = 50 - 2(10) = 30$

(a) $CS = \int_0^{10} (50 - 2Q)dQ - 10(30)$

$= [50Q - Q^2]_0^{10} - 300$

$= [50(10) - (10)^2] - [50(0) - 0^2] - 300$

$= 100$

(b) $PS = 10(30) - \int_0^{10} (10 + 2Q)dQ$

$= 300 - [10Q + Q^2]_0^{10}$

$= 300 - \{[10(10) + (10)^2] - [10(0) + 0^2]\}$

$= 100$

4 (a) $\displaystyle\int_1^8 800t^{1/3}\text{d}t = 800\left[\frac{3}{4}t^{4/3}\right]^8$

$\displaystyle= 800\left[\frac{3}{4}(8)^{4/3} - \frac{3}{4}(0)^{4/3}\right]$

$= 9000$

(b) $\displaystyle\int_1^T 800t^{1/3}\text{d}t = 800\left[\frac{3}{4}t^{4/3}\right]_0^T$

$\displaystyle= 800\left[\frac{3}{4}T^{4/3} - \frac{3}{4}(0)^{4/3}\right]$

$= 600T^{4/3}$

We need to solve

$600T^{4/3} = 48\,600$

that is,

$T^{4/3} = 81$

so

$T = 81^{3/4} = 27$

5 $\displaystyle P = \int_0^{10} 5000\text{e}^{-0.06t}\text{d}t$

$\displaystyle= 5000\int_0^{10} \text{e}^{-0.06t}\text{d}t$

$\displaystyle= 5000\left[-\frac{1}{0.06}\text{e}^{-0.06t}\right]_0^{10}$

$\displaystyle= -\frac{5000}{0.06}(\text{e}^{-0.6} - 1)$

$= \$37\,599.03$

Exercise 6.2 (p. 483)

1 16

2 (a) 4

(b) 0. The graph is sketched in Figure S6.1. Integration gives a positive value when the graph is above the x axis and a negative value when it is below the x axis. In this case there are equal amounts of positive and negative area which cancel out. Actual area is twice that between 0 and 2, so is 8.

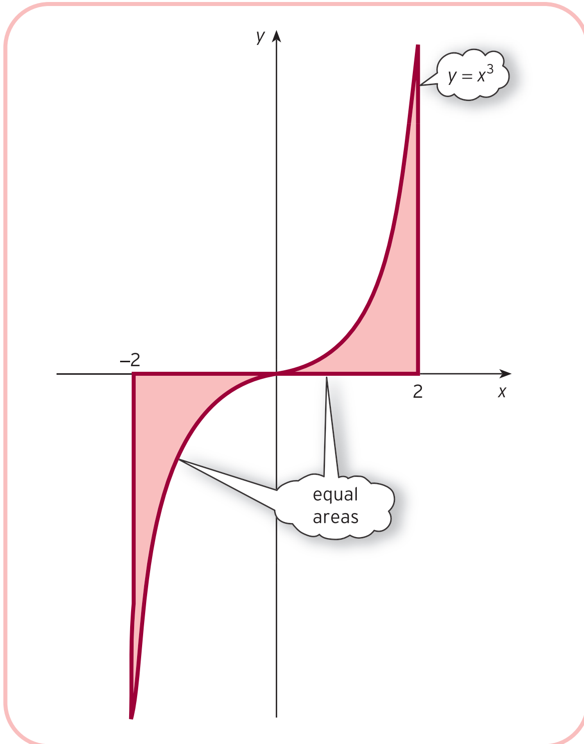


Figure S6.1

3 (a) 100 (b) 20

4 (a) 81 (b) 180

5 (a) 74.67 (b) 58.67

6 \$16 703

7 (a) 12 800 (b) $1600(N^{3/2} - (N-1)^{3/2})$; 4th year

8 \$72 190.14

Exercise 6.2* (p. 484)

1 (a) 27 (b) $\frac{2}{15}$

2 (a) 128 (b) 10

3 $83\frac{1}{3}$ and $130\frac{2}{3}$, respectively.

4 (a) \$427.32 (b) During the 47th year.

5 (a) $\displaystyle\frac{AT^{\alpha+1}}{\alpha+1}$ (b) $\displaystyle\frac{A}{\alpha}(\text{e}^{\alpha T} - 1)$

6 (a) \$2785.84 (b) \$7869.39

(c) \$19 865.24 (d) \$20 000

7 6.9 years.

8 $5x^2 - 2x$

9 $\displaystyle\frac{100S}{r}(1 - \text{e}^{-rn/100})$

10 **(a)** $\frac{1}{2}, \frac{19}{20}, \frac{199}{200}; 1$

(b) $2\sqrt{2}-2,\ 2\sqrt{20}-2,\ 2\sqrt{200}-2$; integral does not exist because these numbers are increasing without bound.

11 **(a)** 1.367 544 468, 1.8, 1.936 754 441; 2.

(b) 9, 99, 999; integral does not exist because these numbers are increasing without bound

Chapter 7

Section 7.1

Practice Problems

1 **(a)** 2×2, 1×5, 3×5, 1×1

(b) 1, 4, 6, 2, 6, ?, 6; the value of c_{43} does not exist, because C has only three rows.

2 $$\mathbf{A}^T = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 4 & 7 & 1 & -5 \\ 0 & 6 & 3 & 1 \\ 1 & 1 & 5 & 8 \\ 2 & 4 & -1 & 0 \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} 1 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

$$\mathbf{C}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \mathbf{C}$$

Matrices with the property that $\mathbf{C}^T = \mathbf{C}$ are called **symmetric.** Elements in the top right-hand corner are a mirror image of those in the bottom left-hand corner.

3 **(a)** $\begin{bmatrix} 1 & 7 \\ 3 & -8 \end{bmatrix}$ **(c)** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **(e)** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Part (b) is impossible because **A** and **C** have different orders.

4 **(1)** **(a)** $\begin{bmatrix} 2 & -4 \\ 6 & 10 \\ 0 & 8 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 \\ 5 & 12 \\ 1 & 10 \end{bmatrix}$ **(d)** $\begin{bmatrix} 2 & -6 \\ 10 & 24 \\ 2 & 20 \end{bmatrix}$

From (a) and (b)

$$2\mathbf{A} + 2\mathbf{B} = \begin{bmatrix} 2 & -4 \\ 6 & 10 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 10 & 24 \\ 2 & 20 \end{bmatrix}$$

which is the same as (d), so

$$2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$$

(2) **(a)** $\begin{bmatrix} 3 & -6 \\ 9 & 15 \\ 0 & 12 \end{bmatrix}$ **(b)** $\begin{bmatrix} -6 & 12 \\ -18 & -30 \\ 0 & -24 \end{bmatrix}$

From (a),

$$-2(3\mathbf{A}) = -2\begin{bmatrix} 3 & -6 \\ 9 & 15 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ -18 & -30 \\ 0 & -24 \end{bmatrix}$$

which is the same as (b), so

$$-2(3\mathbf{A}) = -6\mathbf{A}$$

5 **(a)** [8] because

$$1(0) + (-1)(-1) + 0(1) + 3(1) + 2(2) = 8$$

(b) [0] because $1(-2) + 2(1) + 9(0) = 0$.

(c) This is impossible, because **a** and **d** have different numbers of elements.

6

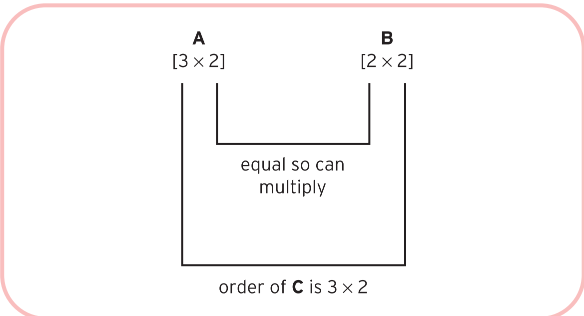


Figure S7.1

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \boxed{1 \quad 2} \\ 0 \quad 1 \\ 3 \quad 1 \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix}1 \\ 3\end{matrix}} & \begin{matrix}2 \\ 4\end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{7} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \boxed{1 \quad 2} \\ 0 \quad 1 \\ 3 \quad 1 \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & \boxed{10} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \quad 2 \\ \boxed{0 \quad 1} \\ 3 \quad 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ \boxed{3} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \quad 2 \\ \boxed{0 \quad 1} \\ 3 \quad 1 \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & \boxed{4} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \quad 2 \\ 0 \quad 1 \\ \boxed{3 \quad 1} \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 \\ \boxed{3} & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \\ \boxed{6} & c_{32} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \quad 2 \\ 0 \quad 1 \\ \boxed{3 \quad 1} \end{bmatrix} \begin{bmatrix} 1 & \boxed{2} \\ 3 & \boxed{4} \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 4 \\ 6 & \boxed{10} \end{bmatrix}$$

7 (a) $\begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 2 & -1 \\ 5 & 5 \end{bmatrix}$ (f) $\begin{bmatrix} 9 & 6 & 13 \\ 27 & 15 & 28 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & 7 & 9 \\ 3 & 3 & 3 \\ 6 & 9 & 12 \end{bmatrix}$ (h) $\begin{bmatrix} 5 & 6 \\ 11 & 15 \end{bmatrix}$

Parts (b), (c) and (e) are impossible because, in each case, the number of columns in the first matrix is not equal to the number of rows in the second.

8 $\mathbf{Ax}$ is the 3×1 matrix

$$\begin{bmatrix} x + 4y + 7z \\ 2z + 6y + 5z \\ 8x + 9y + 5z \end{bmatrix}$$

However, $x + 4y + 7z = -3$, $2x + 6y + 5z = 10$ and $8x + 9y + 5z = 1$, so this matrix is just

$$\begin{bmatrix} -3 \\ 10 \\ 1 \end{bmatrix}$$

which is $\mathbf{b}$. Hence $\mathbf{Ax} \times \mathbf{b}$.

Exercise 7.1 (p. 503)

1 (a) $\mathbf{J} = \begin{bmatrix} 35 & 27 & 13 \\ 42 & 39 & 24 \end{bmatrix}$ $\mathbf{F} = \begin{bmatrix} 31 & 17 & 3 \\ 25 & 29 & 16 \end{bmatrix}$

(b) $\begin{bmatrix} 66 & 44 & 16 \\ 67 & 68 & 40 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 10 & 10 \\ 17 & 10 & 8 \end{bmatrix}$

2 (a) $\begin{bmatrix} 4 & 6 & 2 & 18 \\ 2 & 0 & 10 & 0 \\ 12 & 14 & 16 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 14 & 18 & 12 \\ 4 & 2 & 0 & 10 \\ 12 & 8 & 10 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 20 & 20 & 30 \\ 6 & 2 & 10 & 10 \\ 24 & 22 & 26 & 14 \end{bmatrix}$

(d) Same answer as (c).

3 $4\mathbf{B}$, $(\mathbf{CB})^{\mathrm{T}}$, $\mathbf{CBA}$ are possible with order 2×3, 3×4, 2×4, respectively.

4 (a) $\begin{bmatrix} 5900 \\ 1100 \end{bmatrix}$

Total cost charged to each customer.

(b) $\begin{bmatrix} 13 & 7 & 23 & 22 \\ 3 & 1 & 4 & 5 \end{bmatrix}$

Amount of raw materials used to manufacture each customer's goods.

(c) $\begin{bmatrix} 35 \\ 75 \\ 30 \end{bmatrix}$

Total raw material costs to manufacture one item of each good.

(d) $\begin{bmatrix} 1005 \\ 205 \end{bmatrix}$

Total raw material costs to manufacture requisite number of goods for each customer.

(e) [7000]

Total revenue received from customers.

(f) [1210]

Total cost of raw materials.

(g) [5790]

Profit before deduction of labour, capital and overheads.

5 (1) (a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \\ 5 & 5 \\ 2 & 10 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 5 & 10 \end{bmatrix}$

$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$: that is, 'transpose of the sum is the sum of the transposes'.

(2) (a) $\begin{bmatrix} 1 & 5 \\ 4 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 1 \\ 1 & 5 \\ 4 & 9 \end{bmatrix}$

$(\mathbf{CD})^T = \mathbf{D}^T\mathbf{C}^T$: that is 'transpose of a product is the product of the transposes multiplied in reverse order'.

6 (a) $\mathbf{B} + \mathbf{C} = \begin{bmatrix} 0 & 6 \\ 5 & 2 \end{bmatrix}$

so $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -15 & 24 \\ 5 & 14 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} -7 & 25 \\ 6 & 10 \end{bmatrix}$ and

$\mathbf{AC} = \begin{bmatrix} -8 & -1 \\ -1 & 4 \end{bmatrix}$, so

$\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -15 & 24 \\ 5 & 14 \end{bmatrix}$

(b) $\mathbf{AB} = \begin{bmatrix} -7 & 25 \\ 6 & 10 \end{bmatrix}$, so

$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} 32 & 43 \\ 4 & 26 \end{bmatrix}$

$\mathbf{BC} = \begin{bmatrix} 4 & 11 \\ -4 & 4 \end{bmatrix}$, so

$\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 32 & 43 \\ 4 & 26 \end{bmatrix}$

7 $\mathbf{AB} = [-3]$; $\mathbf{BA} = \begin{bmatrix} 1 & 2 & -4 & 0 \\ 7 & 14 & -28 & 21 \\ 3 & 6 & -12 & 9 \\ -2 & -4 & 8 & -6 \end{bmatrix}$

8 (a) $7x + 5y$

$x + 3y$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

Exercise 7.1* (p. 506)

1 (a), (c) and (f) are possible with orders, 5×2, 3×5, 5×5, respectively.

2 (a) $\mathbf{A}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$, $\mathbf{B}^T = \begin{bmatrix} g & i & k \\ h & j & l \end{bmatrix}$

(b) $\mathbf{AB} = \begin{bmatrix} ag + bi + ck & ah + bj + cl \\ dg + ei + fk & dh + ej + fl \end{bmatrix}$

$\mathbf{B}^T\mathbf{A}^T = \begin{bmatrix} ga + ib + kc & gd + ie + kf \\ ha + jb + lc & hd + je + lf \end{bmatrix}$

(c) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

$(\mathbf{A}^T\mathbf{B}^T\mathbf{C}^T)^T = \mathbf{CBA}$

3 (a) $a_{11} = 137.50$

It is the total weekly profit on T-shirts across all shops in the chain.

(b) $b_{33} = 1327.50$

It is the total weekly profit from Shop C for all three goods.

4 $\begin{bmatrix} 6 & 6 \\ 2 & 11 \\ 3 & 1 \end{bmatrix}$

5 109

6 (a) $\mathbf{AI} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{A}$

Similarly, $\mathbf{IA} = \mathbf{A}$.

(b) $\mathbf{A}^{-1}\mathbf{A}$

$$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, $\mathbf{AA}^{-1} = \mathbf{I}$.

(c) $\mathbf{Ix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}$

7 $\begin{bmatrix} k_2 & k_1 \\ -c & 1-a \end{bmatrix}$

Section 7.2

Practice Problems

1 $|\mathbf{A}| = 6(2) - 4(1) = 8 \neq 0$

so **A** is non-singular and its inverse is given by

$$\frac{1}{8}\begin{bmatrix} 2 & -4 \\ -1 & -6 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ -1/8 & 3/4 \end{bmatrix}$$

$|\mathbf{B}| = 6(2) - 4(3) = 0$

so **B** is singular and its inverse does not exist.

2 We need to solve $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 9 & 1 \\ 2 & 7 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 43 \\ 57 \end{bmatrix}$$

Now

$$\mathbf{A}^{-1} = \frac{1}{61}\begin{bmatrix} 7 & -1 \\ -2 & 9 \end{bmatrix}$$

so

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{61}\begin{bmatrix} 7 & -1 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 43 \\ 57 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

3 In equilibrium, $Q_S = Q_D = Q$, say, so the supply equation becomes

$$P = aQ + b$$

Subtracting aQ from both sides gives

$$P - aQ = b \qquad (1)$$

Similarly, the demand equation leads to

$$P + cQ = d \qquad (2)$$

In matrix notation equations (1) and (2) become

$$\begin{bmatrix} 1 & -a \\ 1 & c \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

The coefficient matrix has an inverse,

$$\frac{1}{c+a}\begin{bmatrix} c & a \\ -1 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \frac{1}{c+a}\begin{bmatrix} c & a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix}$$

that is,

$$P = \frac{cb+ad}{c+a} \quad \text{and} \quad Q = \frac{-b+d}{c+a}$$

The multiplier for Q due to changes in b is given by the (2, 1) element of the inverse matrix so is

$$\frac{-1}{c+a}$$

Given that c and a are both positive it follows that the multiplier is negative. Consequently, an increase in b leads to a decrease in Q.

4 $A_{11} = +\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$

$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$

$A_{13} = +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$

$A_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$A_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{31} = +\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{33} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

5 Expanding along the top row of **A** gives

$$|\mathbf{A}| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(7) + 3(-1) + 3(-1) = 1$$

using the values of A_{11}, A_{12} and A_{13} from Practice Problem 4. Other rows and columns are treated similarly. Expanding down the last column of B gives

$$|\mathbf{B}| = b_{13}B_{13} + b_{23}B_{23} + b_{33}B_{33}$$
$$= 0(B_{13}) + 0(B_{23}) + 0(B_{33}) = 0$$

6 The cofactors of **A** have already been found in Practice Problem 4. Stacking them in their natural positions gives the adjugate matrix

$$\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Transposing gives the adjoint matrix

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The determinant of **A** has already been found in Practice Problem 5 to be 1, so the inverse matrix is the same as the adjoint matrix.

The determinant of **B** has already been found in Practice Problem 5 to be 0, so **B** is singular and does not have an inverse.

7 Using the inverse matrix in Practice Problem 6,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 32 \\ 37 \\ 35 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}$$

Exercise 7.2 (p. 525)

1 (1) (a) $|\mathbf{A}| = -3$

(b) $|\mathbf{B}| = 4$

(c) $\mathbf{AB} = \begin{bmatrix} 4 & 4 \\ 7 & 4 \end{bmatrix}$

so $|\mathbf{AB}| = -12$. These results give $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$: that is, 'determinant of a product is the product of the determinants'.

(2) (a) $\mathbf{A}^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 5/3 & -2/3 \end{bmatrix}$

(b) $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/2 & -1/4 \end{bmatrix}$

(c) $(\mathbf{AB})^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 7/12 & -1/3 \end{bmatrix}$

These results give $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$: that is, 'inverse of a product is the product of the inverses multiplied in reverse order'.

2 $a = -3/2$, $b = -8/3$

3 (a) $x = 1$, $y = -1$

(b) $x = 2$, $y = 2$

4 $\dfrac{1}{25}\begin{bmatrix} -9 & -1 \\ -2 & -3 \end{bmatrix}$; $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$.

5 (a) $50 - 2P_1 + P_2 = -20 + P_1 \Rightarrow 3P_1 - P_2 = 70$

$10 + P_1 - 4P_2 = -10 + 5P_2 \Rightarrow -P_1 + 9P_2 = 20$

(b) Inverse $= \dfrac{1}{26}\begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix}$

$$\frac{1}{26}\begin{bmatrix} 9 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 70 \\ 20 \end{bmatrix} = \begin{bmatrix} 25 \\ 5 \end{bmatrix}$$

Exercise 7.2* (p. 526)

1 $a = \pm 2$, $b = \pm 4$

2 (a) $\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

(b) $\det(\mathbf{A}) = ad - bc$; $\det(\mathbf{B}) = eh - fg$

$$\det(\mathbf{A}) \times \det(\mathbf{B}) = (ad - bc)(eh - fg)$$
$$= adeh - adfg - bceh + bcfg$$
$$\det(\mathbf{AB}) = (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$
$$= acef + adeh + bcfg + bdgh - acef - adfg - bceh - bdgh$$
$$= adeh + bcfg - adfg - bceh$$

(c) **AB** singular; $\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$

$$= 0 \times \det(\mathbf{B}) = 0$$

3 D

4 364

5 7

6 The determinant of **A** is $-10 \neq 0$, so matrix is non-singular.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/10 & 3/10 & -1/2 \\ 3/10 & -1/10 & 1/2 \\ -1/2 & 1/2 & -1/2 \end{bmatrix}$$

It is interesting to notice that because the original matrix **A** is symmetric, so is $\mathbf{A}^{-1}$. The determinant of **B** is 0, so it is singular and does not have an inverse.

7 Commodity market is in equilibrium when $Y = C + I$, so $Y = aY + b + cr + d$, which rearranges as

$$(1 - a)Y - cr = b + d \quad (1)$$

Money market is in equilibrium when $M_S = M_D$, so $M_S^* = k_1Y + k_2r + k_3$, which rearranges as

$$k_1Y + k_2r = M_S^* - k_3 \quad (2)$$

In matrix notation, equations (1) and (2) become

$$\begin{bmatrix} 1-a & -c \\ k_1 & k_2 \end{bmatrix}\begin{bmatrix} Y \\ r \end{bmatrix} = \begin{bmatrix} b+d \\ M_S^* - k_3 \end{bmatrix}$$

Using the inverse of the coefficient matrix,

$$\begin{bmatrix} Y \\ r \end{bmatrix} = \frac{1}{k_2(1-a)+ck_1} \times \begin{bmatrix} k_2 & c \\ -k_1 & 1-a \end{bmatrix}\begin{bmatrix} b+d \\ M_S^* - k_3 \end{bmatrix}$$

$$Y = \frac{k_2(b+d) + c(M_S^* - k_3)}{k_2(1-a)+ck_1}$$

and

$$r = \frac{k_1(b+d) + (1-a)(M_S^* - k_3)}{k_2(1-a)+ck_1}$$

The required multiplier is

$$\frac{\partial r}{\partial M_S^*} = \frac{1-a}{k_2(1-a)+ck_1}$$

Now $1 - a > 0$ since $a < 1$, so numerator is positive. Also $k_2 < 0$, $1 - a > 0$, gives $k_2(1 - a) < 0$ and $c < 0$, $k_1 > 0$ gives $ck_1 < 0$, so the denominator is negative.

8 $a - 1$, which is non-zero provided $a \neq 1$

$$\frac{1}{a-1}\begin{bmatrix} -a & -1 & a \\ 3a-4 & -1 & 3-2a \\ 1 & 1 & -1 \end{bmatrix}$$

9 $\mathbf{A}^{-1} = \frac{1}{-41}\begin{bmatrix} 29 & 11 & 3 \\ 4 & 10 & -1 \\ 9 & 2 & 8 \end{bmatrix}$;

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 8 \end{bmatrix}$$

10 $\mathbf{A}^{-1} = \frac{1}{45-18a}\begin{bmatrix} 8 & 2a-9 & 6-4a \\ -1 & 18-7a & 5a-12 \\ -18 & 9 & 9 \end{bmatrix}$

$a = 2.5$

11 $-95a + 110$; $a = \frac{22}{19}$

12 (a) $aei - afh - dbi + dch + gbf - gce$

Substituting $d = ka$, $e = kb$, $f = kc$ into this gives

$akbi - akch - kabi + kach + gbkc - gckb = 0$

(b) $\frac{1}{\det(\mathbf{A})}\begin{bmatrix} ei-fh & -(bi-ch) & bf-ce \\ -(di-fg) & ai-cg & -(af-cd) \\ -(-dh+eg) & -(ah-bg) & ae-bd \end{bmatrix}$

13 (a) $\frac{1}{1-a+at}\begin{bmatrix} 1 & 1 & a \\ -a(-1+t) & 1 & a \\ t & t & -1+a \end{bmatrix}$

Autonomous consumption multiplier for Y.

(b) $Y = \frac{b + I^* + G^*}{1-a+at}$

$$C = -\frac{-I^*a - aG^* + I^*ta + taG^* - b}{1-a+at}$$

$$T = \frac{bt + I^*t + G^*t}{1-a+at}$$

Section 7.3

Practice Problems

1 (a) By Cramer's rule

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

where

$$\det(\mathbf{A}_2) = \begin{vmatrix} 2 & 16 \\ 3 & -9 \end{vmatrix} = -66$$

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = -22$$

Hence

$$x_2 = \frac{-66}{-22} = 3$$

(b) By Cramer's rule

$$x_3 = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})}$$

where

$$\det(\mathbf{A}_3) = \begin{vmatrix} 4 & 1 & 8 \\ -2 & 5 & 4 \\ 3 & 2 & 9 \end{vmatrix}$$

$$= 4\begin{vmatrix} 5 & 4 \\ 2 & 9 \end{vmatrix} - 1\begin{vmatrix} -2 & 4 \\ 3 & 9 \end{vmatrix} + 8\begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= 4(37) - 1(-30) + 8(-19)$$

$$= 26$$

and

$$\det(\mathbf{A}) = \begin{vmatrix} 4 & 1 & 3 \\ -2 & 5 & 1 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= 4\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= 4(18) - 1(-11) + 3(-19)$$

$$= 26$$

Hence

$$x_3 = \frac{26}{26} = 1$$

2 The variable Y_d is the third, so Cramer's rule gives

$$Y_d = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})}$$

where

$$\mathbf{A}_3 = \begin{bmatrix} 1 & -1 & I^* + G^* & 0 \\ 0 & 1 & b & 0 \\ -1 & 0 & 0 & 1 \\ -t & 0 & T^* & 1 \end{bmatrix}$$

Expanding along the second row gives

$$\det(\mathbf{A}_3) = 1\begin{vmatrix} 1 & I^* + G^* & 0 \\ -1 & 0 & 1 \\ -t & T^* & 1 \end{vmatrix} - b\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ -t & 0 & 0 \end{vmatrix}$$

since along the second row the pattern is '$- + - +$'. Now

$$\begin{vmatrix} 1 & I^* + G^* & 0 \\ -1 & 0 & 1 \\ -t & T^* & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 4 \\ T^* & 1 \end{vmatrix} - (I^* + G^*)\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix}$$

$$= T^* - (I^* + G^*)(-1 + t)$$

(expanding along the first row) and

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ -t & 0 & 0 \end{vmatrix} = -(-1)\begin{vmatrix} -1 & 1 \\ -t & 1 \end{vmatrix} = -1 + t$$

(expanding down the second column).

Hence

$$\det(\mathbf{A}_3) = -T^* - (I^* + G^*)(-1 + t) - b(-1 + t)$$

From the worked example given in the text,

$$\det(\mathbf{A}) = 1 - a + at$$

Hence

$$Y_d = \frac{-T^* - (I^* + G^*)(-1 + t) - b(-1 + t)}{1 - a + at}$$

3 Substituting C_1, M_1 and I_1^* into the equation for Y_1 gives

$$Y_1 = 0.7Y_1 + 50 + 200 + X_1 - 0.3Y_1$$

Also, since $X_1 = M_2 = 0.1Y_2$, we get

$$Y_1 = 0.7Y_1 + 50 + 200 + 0.1Y_2 - 0.3Y_1$$

which rearranges as

$$0.6Y_1 - 0.1Y_2 = 250$$

In the same way, the second set of equations leads to

$$-0.3Y_1 + 0.3Y_2 = 400$$

Hence

$$\begin{bmatrix} 0.6 & -0.1 \\ -0.3 & 0.6 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 250 \\ 400 \end{bmatrix}$$

In this question both Y_1 and Y_2 are required, so it is easier to solve using matrix inverses rather than Cramer's rule, which gives

$$Y_1 = \frac{1}{0.15}\begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.6 \end{bmatrix}\begin{bmatrix} 250 \\ 400 \end{bmatrix} = \frac{1}{0.15}\begin{bmatrix} 115 \\ 315 \end{bmatrix}$$

Hence $Y_1 = 766.67$ and $Y_2 = 2100$. The balance of payments for country 1 is

$$\begin{aligned} X_1 - M_1 &= M_2 - M_1 \\ &= 0.1Y_2 - 0.3Y_1 \\ &= 0.1(2100) - 0.3(766.67) \\ &= -20 \end{aligned}$$

Moreover, since only two countries are involved, it follows that country 2 will have a surplus of 20

Exercise 7.3 (p. 537)

1 **(a)** 1 **(b)** 1 **(c)** 5

2 **(a)** 2 **(b)** −1 **(c)** 1

3 **(a)** $x = 1, \quad y = -1$ **(b)** $x = -2, \quad y = 3$

(c) $x = 7, \quad y = -10$

4 **(a)** $400 - 5P_1 - 3P_2 = -60 + 3P_1 \Rightarrow 8P_1 + 3P_2 = 460$

$300 - 2P_1 - 3P_2 = -100 + 2P_2 \Rightarrow 2P_1 + 5P_2 = 400$

(b) $32\frac{6}{17}$

5 **(a)** $\begin{bmatrix} 1 & -1 \\ -a & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I^* \\ b \end{bmatrix}$

(b) $C = \dfrac{\begin{vmatrix} 1 & I^* \\ -a & b \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -a & 1 \end{vmatrix}} = \dfrac{b + aI^*}{1 - a}$

Exercise 7.3* (p. 538)

1 **(a)** $x_1 = \dfrac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \dfrac{72}{18} = 4$

(b) $x_2 = \dfrac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \dfrac{126}{42} = 3$

(c) $x_4 = \dfrac{\det(\mathbf{A}_4)}{\det(\mathbf{A})} = \dfrac{-1425}{475} = -3$

2 $\dfrac{b - aT^* + a(I^* + G^*)(t - 1)}{1 - a + at}$

3 (a) $\begin{bmatrix} 1 & -1 & 0 \\ -a & 1 & a \\ -t & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I^* + G^* \\ b \\ T^* \end{bmatrix}$

(b) $Y = \dfrac{I^* + G^* + b - aT^*}{1 - a + at}$

4 The equations can be rearranged as

$$Y - C + M = I^* + G^* + X^*$$
$$-aY + C + 0M = b$$
$$-mY + 0C + M = M^*$$

as required.

Autonomous investment multiplier, $\dfrac{1}{1 - a + m}$, is positive because $1 - a$ and m are both positive.

5 The multiplier is

$$\frac{-k_1}{k_2(1 - a) + ck_1}$$

which is positive since the top and bottom of this fraction are both negative. To see that the bottom is negative, note that $k_2(1 - a) < 0$ because $k_2 < 0$ and $a < 1$, and $ck_1 < 0$ because $c < 0$ and $k_1 > 0$.

6 The equations are

$$0.6Y_1 - 0.1Y_2 - I_1^* = 50$$
$$-0.2Y_1 + 0.3Y_2 = 150$$
$$0.2Y_1 - 0.1Y_2 = 0$$

The third equation follows from the fact that if the balance of payments is 0 then $M_1 = X_1$, or equivalently, $M_1 = M_2$. Cramer's rule gives

$$I_1^* = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})} = \frac{4}{0.04} = 100$$

7 $\begin{bmatrix} 1 - a_1 + m_1 & -m_2 \\ -m_1 & 1 - a_2 + m_2 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} b_1 + I_1^* \\ b_2 + I_2^* \end{bmatrix}$

$$Y_1 = \frac{(b_1 + I_1^*)(1 - a_2 + m_2) + m_2(b_2 + I_2^*)}{(1 - a_1 + m_1)(1 - a_2 + m_2) - m_1m_2}$$

The multiplier is

$$\frac{m_2}{(1 - a_1 + m_1)(1 - a_2 + m_2) - m_1m_2}$$

which is positive since the top and bottom of the fraction are both positive. To see that the bottom is positive, note that since $a_1 < 1$, $1 - a_i + m_i > m_i$, so that $(1 - a_1 + m_1)(1 - a_2 + m_2) > m_1m_2$. Hence the national income of one country rises as the investment in the other country rises.

Section 7.4

Practice Problems

1 $\mathbf{d} = \begin{bmatrix} 1000 \\ 300 \\ 700 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0 \end{bmatrix}\begin{bmatrix} 1000 \\ 300 \\ 700 \end{bmatrix} = \begin{bmatrix} 540 \\ 70 \\ 570 \end{bmatrix}$

2 The matrix of technical coefficients is

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}$$

so

$$\mathbf{I} - \mathbf{A} = \begin{bmatrix} 0.8 & -0.2 \\ -0.4 & 0.9 \end{bmatrix}$$

which has inverse

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.64}\begin{bmatrix} 0.9 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$$

Hence

$$\mathbf{x} = \frac{1}{0.64}\begin{bmatrix} 0.9 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} 760 \\ 420 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1000 \end{bmatrix}$$

so total output is 1200 units for engineering and 1000 units for transport.

3 Total outputs for I1, I2, I3 and I4 are found by summing along each row to get 1000, 500, 2000 and 1000, respectively. Matrix of technical coefficients is obtained by dividing the columns of the inter-industrial flow table for these numbers to get

$$\mathbf{A} = \begin{bmatrix} 0 & 0.6 & 0.05 & 0.1 \\ 0.1 & 0 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0.4 \\ 0.3 & 0 & 0.05 & 0 \end{bmatrix}$$

4 $\mathbf{I} - \mathbf{A} = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ -0.1 & 0.9 & -0.1 \\ -0.1 & -0.3 & 0.9 \end{bmatrix}$

so

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.658}\begin{bmatrix} 0.78 & 0.24 & 0.20 \\ 0.10 & 0.79 & 0.11 \\ 0.12 & 0.29 & 0.79 \end{bmatrix}$$

We are given that

$$\Delta\mathbf{d} = \begin{bmatrix} 1000 \\ 0 \\ -800 \end{bmatrix}$$

so

$$\Delta\mathbf{x} = \frac{1}{0.658}\begin{bmatrix} 0.78 & 0.24 & 0.20 \\ 0.10 & 0.79 & 0.11 \\ 0.12 & 0.29 & 0.79 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 0 \\ -800 \end{bmatrix}$$

$$= \begin{bmatrix} 942 \\ 18 \\ -778 \end{bmatrix}$$

Hence, total outputs for I1 and I2 rise by 942 and 18 respectively, and total output for I3 falls by 778 (to the nearest whole number).

Exercise 7.4 (p. 551)

1 $[450 \quad 900 \quad 400 \quad 4350 \quad 50]^T$

2 $\begin{bmatrix} 2 & 1 \\ 1.5 & 2 \end{bmatrix}$

3 **(a)** $[500 \quad 1000]^T$ **(b)** $\begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.5 \end{bmatrix}$

(c) $\frac{1}{36}\begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.8 \end{bmatrix}$ **(d)** $[694 \quad 1056]^T$

Exercise 7.4* (p. 552)

1 **(a)** $\mathbf{A} = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.25 \end{bmatrix}$

(b) $\begin{bmatrix} 200 \\ 600 \end{bmatrix}$

2 $\mathbf{A} = \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0 & 0.1 \\ 0.2 & 0.2 & 0 \end{bmatrix}$

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.924}\begin{bmatrix} 0.98 & 0.14 & 0.21 \\ 0.12 & 0.96 & 0.12 \\ 0.22 & 0.22 & 0.99 \end{bmatrix}$$

(a) 1000 units of water, 500 units of steel and 1000 units of electricity.

(b) The element in the first row and third column of $(\mathbf{I} - \mathbf{A})^{-1}$ is

$$\frac{0.21}{0.924} = 0.23$$

Change in water output is $0.23 \times 100 = 23$.

3 **(a)** $\mathbf{A} = \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.1 \end{bmatrix}$

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.216}\begin{bmatrix} 0.48 & 0.24 & 0.24 \\ 0.39 & 0.60 & 0.33 \\ 0.30 & 0.24 & 0.42 \end{bmatrix}$$

(b) 100 000, 162 500, 125 000

(c) 278 (nearest unit)

4 **(a)** $\mathbf{BC} = \begin{bmatrix} 0.239 & 0 & 0 \\ 0 & 0.239 & 0 \\ 0 & 0 & 0.239 \end{bmatrix}$

(b) $\begin{bmatrix} -510 \\ -870 \\ -1930 \end{bmatrix}$

Additional Topic 1

Section T.1.1

Practice Problems

1 The line $-x + 3y = 6$ passes through (0, 2) and (−6, 0). Substituting $x = 1$, $y = 4$ into the equation gives

$$-1 + 3(4) = 11$$

This is greater than 6, so the test point satisfies the inequality. The corresponding region is shown in Figure ST1.1.

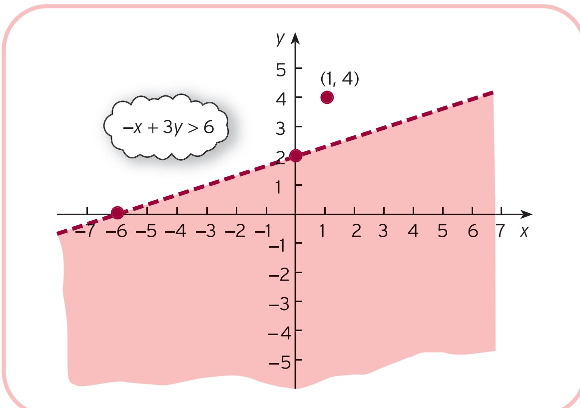


Figure ST1.1

2 The non-negativity constraints indicate that we restrict our attention to the positive quadrant.

The line $x + 2y = 10$ passes through (0, 5) and (10, 0).

The line $3x + y = 10$ passes through (0, 10) and ($^{10}/_3$, 0).

Also the test point (0, 0) satisfies both of the corresponding inequalities, so we are interested in the region below both lines as shown in Figure ST1.2.

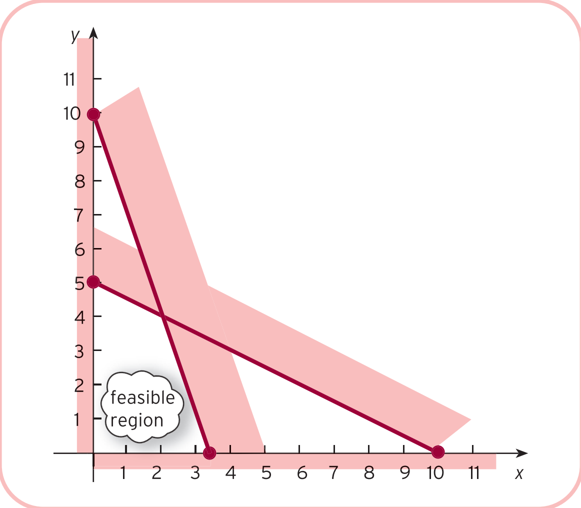


Figure ST1.2

3 The answers to parts (a) and (b) are shown in Figure ST1.3.

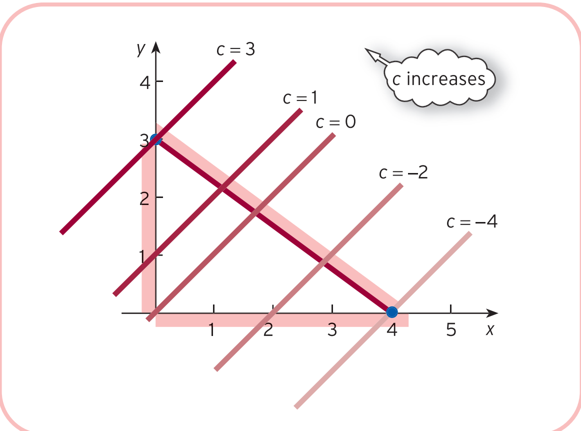


Figure ST1.3

(c) Once c becomes greater than 3, the lines no longer intersect the feasible region. The maximum value of c (that is, the objective function) is therefore 3, which occurs at the corner (0, 3), when $x = 0$, $y = 3$.

4 Step 1

The feasible region is sketched in Figure ST1.4.

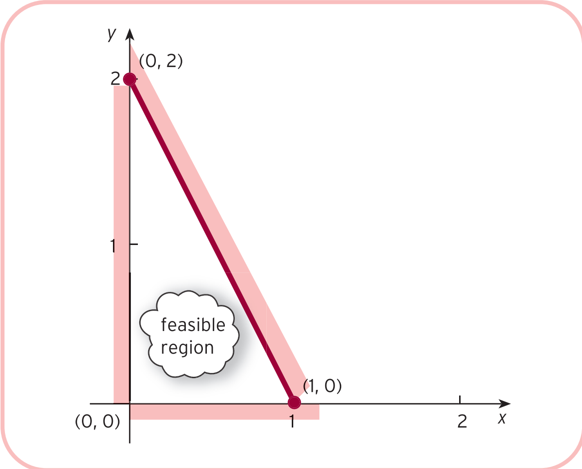


Figure ST1.4

Step 2

Corners are (0, 0), (1, 0) and (0, 2).

Step 3

Corner	Objective function
(0, 0)	0 − 0 = 0
(1, 0)	1 − 0 = 1
(0, 2)	0 − 2 = −2

Minimum is −2, which occurs at (0, 2).

5 Step 1

The feasible region is sketched in Figure ST1.2.

Step 2

Corners are (0, 0), (0, 5), (2, 4) and ($^{10}/_3$, 0).

Step 3

Corner	Objective function
(0, 0)	3(0) + 5(0) = 0
(0, 5)	3(0) + 5(5) = 25
(2, 4)	3(2) + 5(4) = 26
($^{10}/_3$, 0)	3($^{10}/_3$) + 5(0) = 25

Maximum is 26, which occurs at (2, 4).

Exercise T.1.1 (p. 587)

1 (1, 1), (1, −1), (−1, −1), (2, −1), (−2, −1)

2 6

3 The feasible regions for parts (a), (b) and (c) are sketched in Figures ST1.5, ST1.6 and ST1.7, respectively.

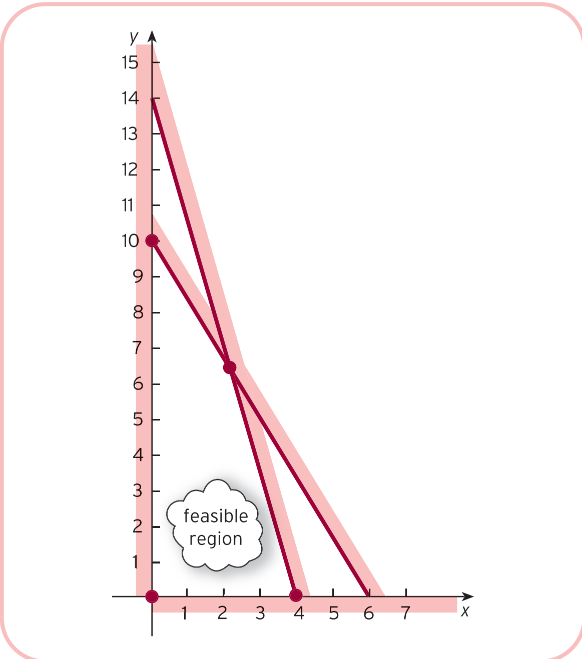


Figure ST1.5

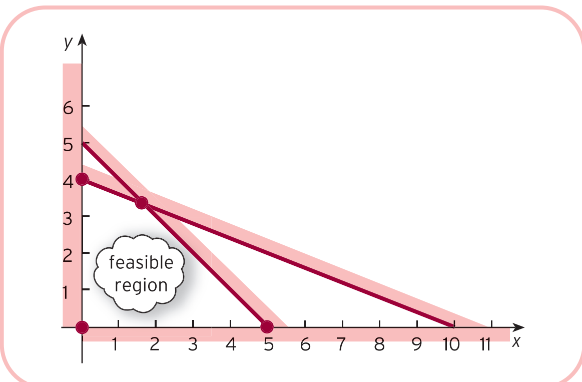


Figure ST1.6

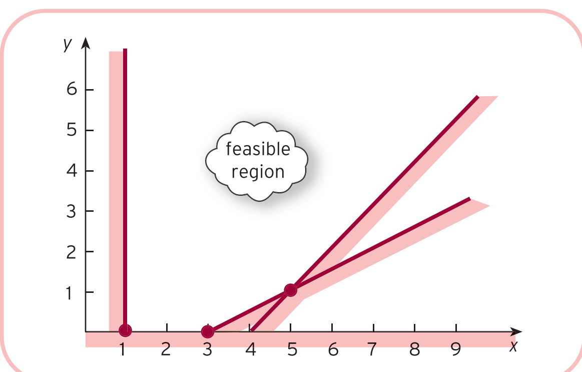


Figure ST1.7

4 **(a)** Maximum is 90, which occurs at (0, 10).

(b) Maximum is 25, which occurs at $(^5/_3, ^{10}/_3)$. Note that the exact coordinates can be found by solving the simultaneous equations

$$2x + 5y = 20$$

$$x + y = 5$$

using an algebraic method.

(c) Minimum is 1, which occurs at (1, 0).

5 Figure ST1.8 shows that the problem does not have a finite solution. The lines $x + y = c$ pass through $(c, 0)$ and $(0, c)$. As c increases, the lines move across the region to the right without bound.

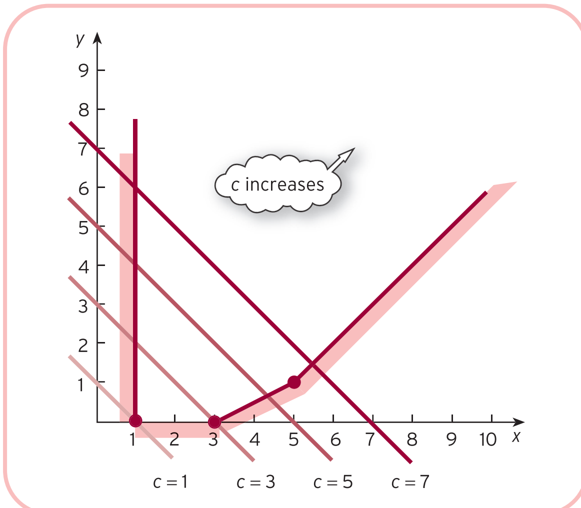


Figure ST1.8

6 No solution

7 **(a)** Unique solution at (2, 0).

(b) Unique solution at (0, 2).

(c) Infinitely many solutions.

Exercise T.1.1* (p. 589)

1 –3.8

2 A

3 **(a)** Maximum is 16, which occurs at (2, 4).

(b) Maximum is 12, which occurs at any point on the line joining (0, 3) and (1, 5).

4 **(a)** There is no feasible region, since the constraints are contradictory.

(b) The feasible region is unbounded and there is no limit to the values that the objective function can take in this region.

5 Minimum is −16, which occurs at the two corners (2, 2) and ($^8/_3$, 0), so any point on the line segment joining these two corners is also a solution.

6 $\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix}$ $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

7 **(a)** **(i)** Line 1: passing through (8, 0) and (0, 16).

Line 2: passing through (12, 0) and (0, 8).

Line 3: passing through (0, 12) and, for example, (8, 20).

Shade under the $y = 0$, line 1, line 2, but above line 3.

Corners: (12, 0), (6, 4)

For the third corner, solve the simultaneous equations

$$2x + y = 16$$
$$-x + y = 12$$

to get ($1^1/_3$, $13^1/_3$)

(ii)

Corners	Objective function
(12, 0)	12
(6, 4)	10
($1^1/_3$, $13^1/_3$)	$14^2/_3$

Optimal point (6, 4)

(iii) $x \geq 0$

(b) **(i)** No solution since $x + y$ increases without bound as the lines $x + y = c$ sweep across the region to the right.

(ii)

Corners	Objective function
(12, 0)	24
(6, 4)	16
($1^1/_3$, $13^1/_3$)	16

Infinitely many solutions; any point on the line segment between (6, 4) and ($1^1/_3$, $13^1/_3$) will be a solution.

(c) The line $2x + 3y = 24$ rearranges as $y = 8 - \frac{2}{3}x$,

so has slope, $-\frac{2}{3}$

The line $ax + 2y = c$ rearranges as $y = \frac{c}{2} - \frac{a}{2}x$,

so has slope, $-\frac{a}{2}$

$$\frac{a}{2} \leq \frac{2}{3} \Rightarrow a \leq \frac{4}{3}$$

Section T.1.2

Practice Problems

1 Let x = weekly output of model COM1,

y = weekly output of model COM2.

Maximize $600x + 700y$ (profit)

subject to

$1200x + 1600y \leq 40\ 000$	(production costs)
$x + y \leq 30$	(total output)
$x \geq 0,\ y \geq 0$	(non-negativity constraints)

The feasible region is sketched in Figure ST1.9.

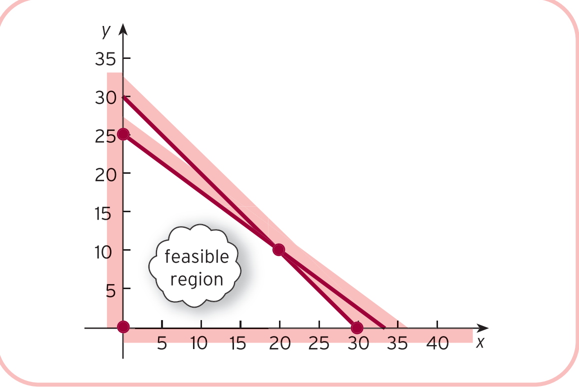


Figure ST1.9

Corner	Profit ($)
(0, 0)	0
(0, 25)	17 500
(20, 10)	19 000
(30, 0)	18 000

The firm should produce 20 computers of model COM1 and 10 of model COM2 to achieve a maximum profit of $19 000.

2 Let x = number of copies of *Microeconomics*,

y = number of copies of *Macroeconomics*.

Maximize $12x + 18y$ (profit)

subject to

$12x + 15y \leq 600$	(printing time)
$18x + 9y \leq 630$	(binding time)
$x \geq 0,\ y \geq 0$	(non-negativity constraints)

The feasible region is sketched in Figure ST1.10.

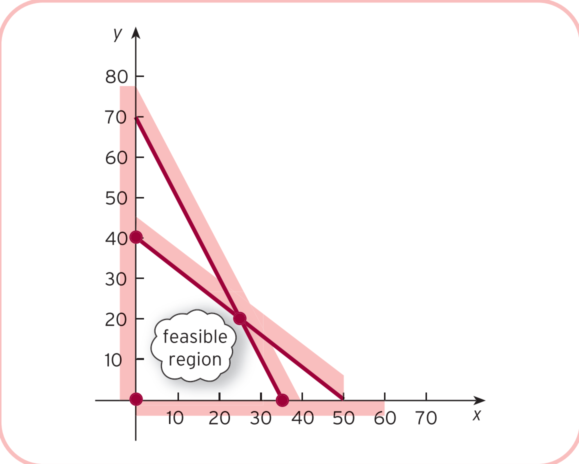


Figure ST1.10

The publisher should produce 40 copies of *Macroeconomics* and no copies of *Microeconomics* to achieve a maximum profit of \$720.

3 Maximize $3x + 7y$ (utility)

subject to

$150x + 70y \leq 2100$ (cost)

$x \geq 9, y \geq 0$

The feasible region is sketched in Figure ST1.11.

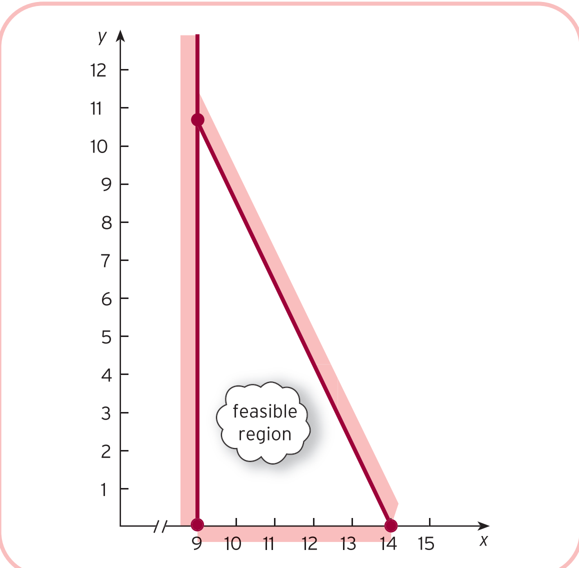


Figure ST1.11

Corner	Objective function
(9, 0)	27
(14, 0)	42
$(9, ^{75}/_7)$	102

The maximum value of U occurs at $(9, ^{75}/_7)$. However, it is impossible to visit the theatre $^{75}/_7$ times. The point in the feasible region with whole-number coordinates which maximizes utility is (9, 10), so we need to buy 9 items of clothing and visit the theatre 10 times per year.

Exercise T.1.2 (p. 603)

1 The manufacturer should produce 10 bikes of type B and 15 of type C each month to achieve a maximum profit of \$5100.

2 The firm should produce 720 cartons of 'The Caribbean' and 630 cartons of 'Mr Fruity' each week to give a maximum profit of \$650.70.

3 The student should order a quarterpounder served with 6 oz chips to consume a minimum of 860 calories. Note that the unbounded feasible region causes no difficulty here, because the problem is one of minimization.

4 **(a)** $12\,000x + 15\,000y$; maximize.

(b) Total number of students is $x + y$ and this must not exceed 9000, so

$$x + y \leq 9000$$

For example, at least $^3/_4$ of the students are US citizens, so

$$\frac{3}{4}(x + y) \leq x \quad \Rightarrow \quad 3x + 3y \leq 4x \quad \Rightarrow \quad x \geq 3y$$

All non-US students together with a quarter of the US students must be given residential places, so

$$y + \frac{1}{4}x \leq 5000$$

Any slack would be automatically taken up by US students since these are the only students left.

$$x \geq 0; \ y \geq 0$$

(c) x and y must be whole numbers.

Exercise T.1.2* (p. 604)

1 (1) and (6).

2 **(a)** The firm should make 30 jackets and 6 pairs of trousers each week to achieve a maximum profit of \$444.

(b) The profit margin on a pair of trousers should be between \$8 and \$14.

3 The optimal diet consists of 1.167 kg of fish meal and 1.800 kg of meat scraps, which gives a minimum cost of \$1.69 per pig per day.

4 $x = 40$, $y = 0$, $z = 100$; don't forget to type in the command `with (simplex):`.

5 x_1 = number of hectares for barley in large field in year 1

x_2 = number of hectares for barley in small field in year 1

x_3 = number of hectares for cattle in large field in year 1

x_4 = number of hectares for cattle in small field in year 1

x_5, x_6, x_7, x_8 denote corresponding areas for year 2

Maximize $400x_1 + 220x_2 + 350x_3 + 200x_4 + 420x_5 + 240x_6 + 540x_7 + 320x_8$

subject to

$$x_1 + x_3 \le 1400$$
$$x_2 + x_4 \le 800$$
$$x_5 + x_7 \le 1400$$
$$x_6 + x_8 \le 800$$
$$2x_1 + 2x_2 - x_3 - x_4 \ge 0$$
$$2x_5 + 2x_6 - x_7 - x_8 \ge 0$$
$$6x_3 + 6x_4 - 5x_7 - 5x_8 \ge 0$$

together with the eight non-negativity constraints, $x_i \ge 0$.

$$x_1 = \frac{8800}{9}, x_2 = 0, x_3 = \frac{3800}{9}, x_4 = 800,$$

$$x_5 = 0, x_6 = \frac{2200}{3}, x_7 = 1400, x_8 = \frac{200}{3}$$

Additional Topic 2

Section T.2.1

Practice Problems

1 **(1)** **(a)** 1, 3, 9, 27; 3^t

(b) 7, 21, 63, 189; $7(3^t)$

(c) A, $3A$, $9A$, $27A$; $A(3^t)$

(2) **(a)** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}; \left(\frac{1}{2}\right)^t$

(b) $7, 7\left(\frac{1}{2}\right), 7\left(\frac{1}{4}\right), 7\left(\frac{1}{8}\right); 7\left(\frac{1}{2}\right)^t$

(c) $A, A\left(\frac{1}{2}\right), A\left(\frac{1}{4}\right), A\left(\frac{1}{8}\right); A\left(\frac{1}{2}\right)^t$

(3) $A, Ab, Ab^2, Ab^3; A(b^t)$

2 **(a)** The complementary function is the solution of

$$Y_t = -\frac{1}{2}Y_{t-1}$$

so is given by

$$CF = A\left(-\frac{1}{2}\right)^t$$

For a particular solution we try

$$Y_t = D$$

Substituting this into

$$Y_t = -\frac{1}{2}Y_{t-1} + 6$$

gives

$$D = -\frac{1}{2}D + 6$$

which has solution $D = 4$, so

$$PS = 4$$

The general solution is

$$Y_t = A\left(-\frac{1}{2}\right)^t + 4$$

The initial condition, $Y_0 = 0$, gives

$$0 = A + 4$$

so A is -4. The solution is

$$Y_t = -4\left(-\frac{1}{2}\right)^t + 4$$

From the staircase diagram shown in Figure ST2.1 we see that Y_t oscillates about $Y_t = 4$. Moreover, as t increases, these oscillations damp down and Y_t converges to 4. Oscillatory convergence can be expected for any solution

$$Y_t = A(b^t) + PS$$

when $-1 < b < 0$.

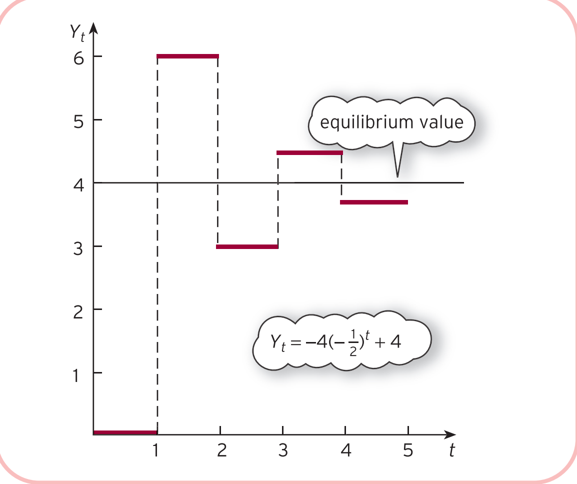


Figure ST2.1

(b) $CF = A(-2)^t$ and $PS = 3$ so $Y_t = A(-2)^t + 3$. Initial condition gives $A = 1$, so $Y_t = (-2)^t + 3$. From Figure ST2.2 we see that Y_t oscillates about 3 and that these oscillations explode with increasing t. Oscillatory divergence can be expected for any solution

$$Y_t = A(b^t) + PS$$

when $b < -1$.

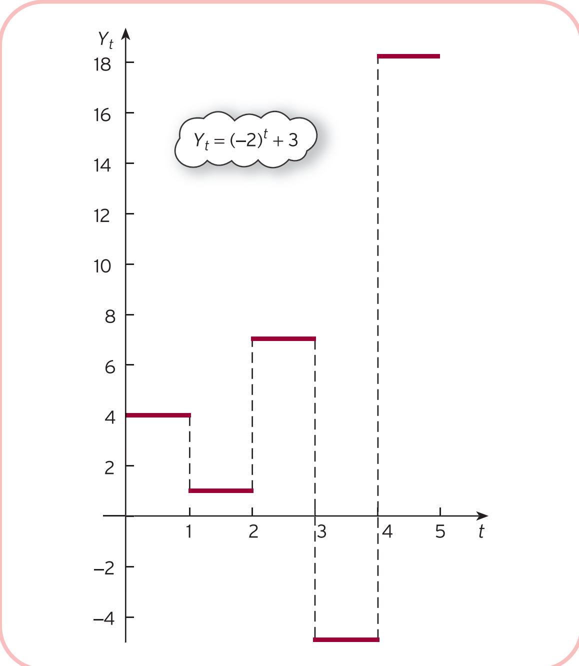


Figure ST2.2

3 $Y_t = C_t + I_t$

$= 0.9Y_{t-1} + 250 + 350$

$= 0.9Y_{t-1} + 600$

This has solution

$Y_t = A(0.9)^t + 6000$

The initial condition, $Y_0 = 6500$, gives $A = 500$, so

$Y_t = 500(0.9)^t + 6000$

The system is stable because $-1 < 0.9 < 1$. In fact, Y_t converges uniformly to the equilibrium value, 6000.

4 $-2P_t + 22 = P_{t-1} - 8$

rearranges to give

$$P_t = -\frac{1}{2}P_{t-1} + 15$$

so has solution

$$P_t = A\left(-\frac{1}{2}\right)^t + 10$$

The initial condition, $P_0 = 11$, gives $A = 1$, so

$$P_t = \left(-\frac{1}{2}\right)^t + 10$$

From the demand equation,

$Q_t = -2P_t + 22$

we have

$$Q_t = -2\left[\left(-\frac{1}{2}\right)^t + 10\right] + 22 = -2\left(-\frac{1}{2}\right)^t + 2$$

The system is stable because $-1 < -{}^1\!/_2 < 1$. In fact, P_t and Q_t display oscillatory convergence and approach the equilibrium values of 2 and 10 respectively as t increases.

5 $-2P_t + 80 = 3P_{t-1} - 20$

rearranges to give

$P_t = -1.5P_{t-1} + 50$

so has solution

$P_t = A(-1.5)^t + 20$

The initial condition, $P_0 = 8$, gives $A = -12$, so

$P_t = -12(-1.5)^t + 40$

From the demand equation

$Q_t = -2P_t + 80$

$= -2[-12(-1.5)^t + 20] + 80$

$= 24(-1.5)^t + 4$

The system is unstable because $-1.5 < -1$. In fact, P_t and Q_t display oscillatory divergence as t increases.

Exercise T.2.1 (p. 622)

1 **(a)** $Y_0 = 0$, $Y_1 = 2 = 2 \times 1$, $Y_2 = 4 = 2 \times 2$, $Y_3 = 6 = 2 \times 3$, . . .

Hence $Y_t = 2t$ and displays uniform divergence as shown in Figure ST2.3.

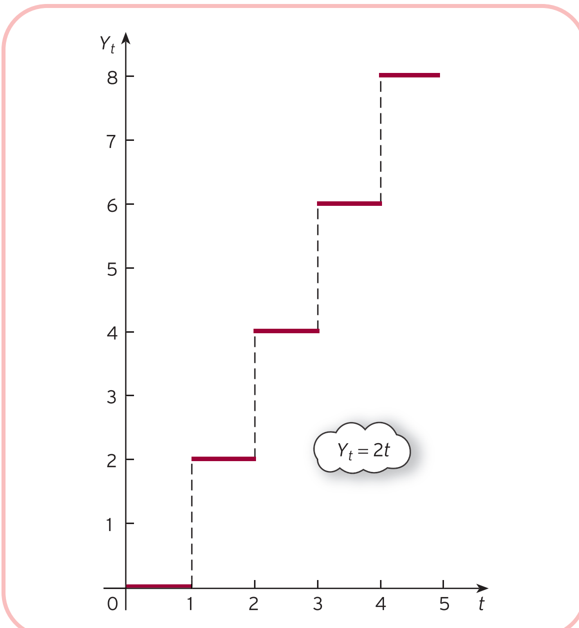


Figure ST2.3

(b) $Y_0 = 4$, $Y_1 = 2$, $Y_2 = 4$, $Y_3 = 2$, . . .

So Y_t is 4 when t is even and 2 when t is odd. Hence Y_t oscillates with equal oscillations as shown in Figure ST2.4.

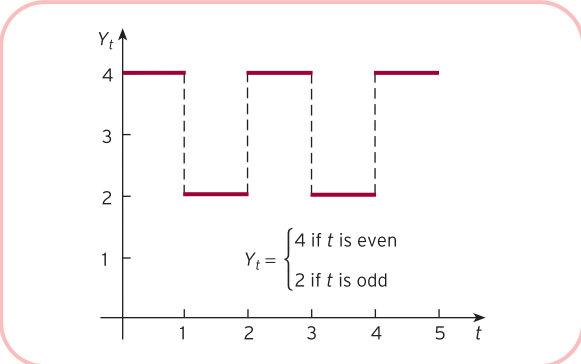


Figure ST2.4

(c) $Y_0 = 3, Y_1 = 3, Y_2 = 3, Y_3 = 3, \ldots$

Hence $Y_t = 3$ for all t and remains fixed at this value.

2 **(a)** $Y_t = -7\left(\frac{1}{4}\right)^t + 8$; uniform convergence to 8.

(b) $Y_t = (-4)^t + 1$; oscillatory divergence.

3 $Y_t = 500(0.8)^t + 2500$; stable.

4 $P_t = 10(-0.5)^t + 60$; stable.

5 3100

6 Substitute assumptions (1) and (2) into (3) to get

$$\beta(Y_t - Y_{t-1}) = \alpha Y_t$$

which rearranges as

$$Y_t = \left(\frac{\beta}{\beta - \alpha}\right)Y_{t-1}$$

with solution

$$Y_t = \left(\frac{\beta}{\beta - \alpha}\right)^t Y_0$$

If $\alpha = 0.1$ and $\beta = 1.4$ then $Y_t = (1.08)^t Y_0$.

As t increases, Y_t diverges uniformly, so unstable.

Exercise T.2.1* (p. 623)

1 Uniform convergence.

2 **(a)** −1, 1/2, 2, −1; 1/2 **(b)** 7, 11, 15, 19; $Y_t = 4t + 3$

3 $Y_t = \dfrac{c(1 - b^t)}{1 - b} + ab^t$

4 $Y_t = 4000 + 400n$; so unstable.

5 **(a)** $aP_{t-1} - b = -cP_t + d$

$$\Rightarrow -cP_t = aP_{t-1} - (b + d) \Rightarrow P_t = -\frac{a}{c}P_{t-1} + \frac{b + d}{c}$$

(b) $P_{t-1} = P_t = D$

$$D = -\frac{aD}{c} + \frac{b + d}{c} \Rightarrow (a + c)D = b + d \Rightarrow D = \frac{b + d}{a + c}$$

$$P_t = A\left(-\frac{a}{c}\right)^t + \frac{b + d}{a + c}$$

(c) $a < c$

$$P = \frac{b + d}{a + c}$$

$$Q = a\left(\frac{b + d}{a + c}\right) - b = \frac{a(b + d) - b(a + c)}{a + c} = \frac{ad - bc}{a + c}$$

6 $P_t = (1 - ae - ce)P_{t-1} + e(b + d)$

7 **(a)** $\text{CF} = A(0.1)^t$

(b) $\text{PS} = 6(0.6)^t$

(c) $Y_t = A(0.1)^t + 6(0.6)^t$

$Y_t = 3(0.1)^t + 6(0.6)^t$

(d) Stable.

8 **(a)** $\text{CF} = A(0.2)^t$

(b) $\text{PS} = t + 6$

(c) $Y_t = A(0.2)^t + t + 6,$

$Y_t = 4(0.2)^t + t + 6$

(d) Unstable.

9 **(a)** $Y_t = 0.6Y_{t-1}^{0.8} + 160$

(b) 163.8, 195.4, 200.8, 201.7, 201.9, 201.9, 201.9, 201.9; stable.

(c) No.

10 **(a)** $k_{t+1} = 0.99k_t + 0.2k_t^{0.6}$

(b) 1789; uniform; same behaviour.

Section T.2.2

Practice Problems

1 **(a)** $5e^{5t}$ **(b)** $-6e^{-3t}$ **(c)** mAe^{mt}

2 **(a)** The function that differentiates to 4 times itself is $y = Ae^{4t}$. The condition $y(0) = 6$ gives $A = 6$, so the solution is $y = 6e^{4t}$.

(b) The function that differentiates to −5 times itself is $y = Ae^{-5t}$. The condition

$y(0) = 2$ gives $A = 2$, so the solution is $y = 2e^{-5t}$.

3 The complementary function is the solution of

$$\frac{dy}{dt} = 3y$$

and is given by

$$\text{CF} = Ae^{3t}$$

For a particular solution we try a constant function

$$y(t) = D$$

Substituting this into the original equation,

$$\frac{dy}{dt} = 3y - 60$$

gives

$$0 = 3D - 60$$

which has solution $D = 20$. The general solution is therefore

$$y(t) = Ae^{3t} + 20$$

Finally, substituting $t = 0$ gives

$$y(0) = A + 20 = 30$$

and so A is 10. Hence

$$y(t) = 10e^{3t} + 20$$

A graph of y against t is sketched in Figure ST2.5, which indicates that $y(t)$ rapidly diverges. We would expect divergence to occur for any solution

$$y(t) = Ae^{mt} + D \quad (A \neq 0)$$

when $m > 0$.

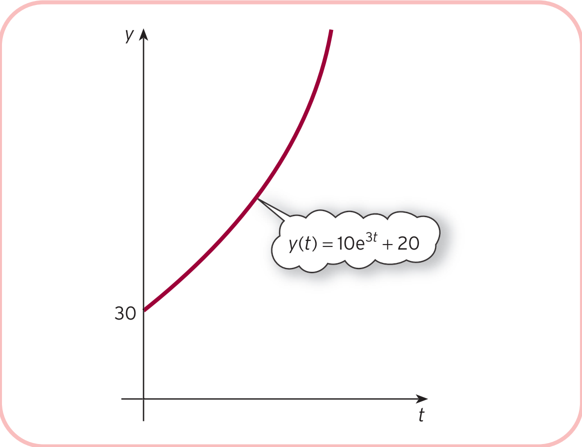


Figure ST2.5

4 Substituting the expressions for C and I into

$$\frac{dY}{dt} = 0.1(C + I - Y)$$

gives

$$\frac{dY}{dt} = 0.1(0.9Y + 100 + 300 - Y) = -0.01Y + 40$$

The complementary function is given by

$$CF = Ae^{-0.01t}$$

and for a particular solution we try a constant function

$$Y(t) = D$$

Substituting this into the differential equation gives

$$0 = -0.01D + 40$$

which has solution $D = 4000$. The general solution is therefore

$$Y(t) = Ae^{-0.01t} + 4000$$

The initial condition, $Y(0) = 2000$, gives

$$A + 4000 = 2000$$

and so A is −2000. Hence

$$Y(t) = -2000e^{-0.01t} + 4000$$

This system is stable because the complementary function is a negative exponential and so $Y(t)$ converges to its equilibrium value of 4000 as t increases.

5 Substituting the expressions for Q_S and Q_D into

$$\frac{dP}{dt} = \tfrac{1}{3}(Q_D - Q_S)$$

gives

$$\frac{dP}{dt} = \tfrac{1}{3}[(-P + 4) - (2P - 2)] = -P + 2$$

The complementary function is given by

$$CF = Ae^{-t}$$

and for a particular solution we try a constant function

$$P(t) = D$$

Substituting this into the differential equation gives

$$0 = -D + 2$$

which has solution $D = 2$. The general solution is therefore

$$P(t) = Ae^{-t} + 2$$

The initial condition, $P(0) = 1$, gives

$$A + 2 = 1$$

and so A is −1. Hence

$$P(t) = -e^{-t} + 2$$

From the supply and demand equations

$$Q_S(t) = 2P - 2 = 2(-e^{-t} + 2) - 2 = -2e^{-t} + 2$$

$$Q_D(t) = -P + 4 = -(-e^{-t} + 2) + 4 = e^{-t} + 2$$

All three functions involve a negative exponential, so the system is stable.

Exercise T.2.2 (p. 640)

1 **(a)** $t^2 + 7$ **(b)** $-\tfrac{1}{3}e^{-3t} + \tfrac{1}{3}$ **(c)** $\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 5t + 1$

2 **(a)** $-20e^{-3t} + 60$; starting at 40, $y(t)$ increases uniformly to 60.

(b) $20e^{-3t} + 60$; starting at 80, $y(t)$ decreases uniformly to 60.

(c) 60; $y(t)$ remains at the equilibrium level of 60 for all time.

3 \$202.04

4 $Y(t) = 5000e^{-0.05t} + 10\,000$; stable.

5 $Y(t) = 2000e^{0.15t} - 1800$; unstable.

6 $P(t) = -e^{-2.5t} + 2$; $Q_S(t) = -3e^{-2.5t} + 5$; $Q_D(t) = 2e^{-2.5t} + 5$; stable.

Exercise T.2.2* (p. 641)

1 $t^3 - 8\sqrt{t} + 4$

2 $\frac{1}{3}(23 + 4e^{-2.4t})$

3 **(a)** $S = 4000e^{0.06t}$ **(b)** Continuous.

4 $y = 14 - 4e^{-2t}$; graph has a y-intercept of 10, increases and approaches 14.

5 1200

6 **(a)** $Y = 4400e^{-0.04t} + 3600$

(b) $S = 880e^{-0.04t} + 300$

(c) $t = 52$; $\frac{dY}{dt} = -22$

7 0

8 Substitute assumptions (1) and (3) into (2) to get

$$\beta\frac{dY}{dt} = \alpha Y$$

which rearranges as

$$\frac{dY}{dt} = \frac{\alpha}{\beta}Y$$

with solution

$$Y(t) = Y(0)e^{(\alpha/\beta)t}$$

This is unstable because $\alpha/\beta > 0$.

9 The right-hand side is

$$my + c = m\left(Ae^{mt} - \frac{c}{m}\right) + c$$
$$= Ame^{mt} - c + c = Ame^{mt}$$

which we recognize as the derivative of $y(t)$.

10 **(a)** Ae^{-2t} **(b)** e^{3t} **(c)** $Ae^{-2t} + e^{3t}$; $6e^{-2t} + e^{3t}$

(d) Unstable.

11 **(a)** Ae^{-t} **(b)** $4t - 7$

(c) $y = Ae^{-t} + 4t - 7$; $y = 8e^{-t} + 4t - 7$

(d) Unstable.

12 $y(t) = \sqrt{(t^2 + 1)}$. For large t, the solution is well approximated by a straight line with equation $y(t) = t$.

13 **(a)** $y(t) = 0.5\left(\frac{t+4}{t+2}\right)$

(b) $y(t) = 0.5\left(\frac{t-2}{t-4}\right)$

(c) In (a) the fund decreases and approaches the equilibrium value of \$500 000; in (b) the fund decreases and is completely exhausted after 2 years.

14 $K(t)\frac{(ac - b)e^{a^2/2} + b}{a}$

(a) Capital remains constant, at the value c, for all time.

(b) Capital grows without bound.

(c) Capital decreases to zero.

Appendix 1

Practice Problems

1 Slope of chord $= \frac{(5 + \Delta x)^2 - 5^2}{\Delta x}$

$$= \frac{25 + 10\Delta x + (\Delta x)^2 - 25}{\Delta x}$$
$$= 10 + \Delta x$$

Slope of tangent $= \lim_{\Delta x \to 0}(10 + \Delta x) = 10$

2 **(a)** Slope of chord

$$= \frac{[4(x + \Delta x)^2 - 9(x + \Delta x) + 1] - (4x^2 - 9x + 1)}{\Delta x}$$
$$= 8x + 4\Delta x - 9$$

Slope of tangent $= \lim_{\Delta x \to 0}(8x + 4\Delta x - 9) = 8x - 9$

(b) $f(x + \Delta x) - f(x) = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}$

$$= \frac{x^2 - (x + \Delta x)^2}{(x + \Delta x)^2 x^2}$$
$$= \frac{-2x\Delta x - (\Delta x)^2}{(x + \Delta x)^2 x^2}$$

Slope of chord $= \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}$

Slope of tangent $= \lim_{\Delta x \to 0}\left(\frac{-2x - \Delta x}{(x + \Delta x)^2 x^2}\right) = \frac{-2}{x^3}$

3 **(a)** $(a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2)$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

(b) Slope of chord $= \frac{(x + \Delta x)^3 + x^3}{\Delta x}$

$$= 3x^2 + 3x(\Delta x) + (\Delta x)^2$$

Slope of tangent

$$= \lim_{\Delta x \to 0}(3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

Appendix 2

Practice Problems

1 **(a)** $2 \times 1^2 + 3 \times 2^2 = 2 + 12 = 14$

(b) $4x + 6y\frac{dy}{dx} = 0$

$$6y\frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{-4x}{6y} = \frac{2x}{3y}$$

so when $x = 1$, $y = 2$,

$$\frac{dy}{dx} = -\frac{1}{3}$$

2 (a) $-\dfrac{x}{y}$ (b) $-\dfrac{1+6x^2}{3y}$ (c) $-\dfrac{e^{x-y}}{2}$

(d) $\dfrac{(1-e^x)y}{e^x - x - 2y}$ (e) $\dfrac{2x+2y^2}{3-4xy}$ (f) $-(1+x+y)$

Appendix 3

Practice Problems

1 $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

The first principal minor is $2 > 0$.

The second principal minor is $2 \times 2 - 0 \times 0 = 4 > 0$.

Hence the stationary point is a minimum.

2 $\mathbf{H} = \begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$

Principal minors are $-4 < 0$ and $4 > 0$, respectively, so maximum.

3 $\mathbf{H} = \begin{bmatrix} -4L^{-3/2} & 0 \\ 0 & -6K^{-3/2} \end{bmatrix}$

assuming that L is chosen as the first variable.

At the stationary point,

$$\mathbf{H} = \begin{bmatrix} -1/16 & 0 \\ 0 & -1/288 \end{bmatrix}$$

Principal minors are $-1/16 < 0$ and $1/4608 > 0$, respectively, so maximum.

4 $\bar{\mathbf{H}} = \begin{bmatrix} -2 & 2 & -1 \\ 2 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ has determinant $8 > 0$, so maximum.

5 $\bar{\mathbf{H}} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ has determinant $-6 < 0$, so minimum.